TRANSONIC SHOCKS IN MULTIDIMENSIONAL DIVERGENT NOZZLES

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Abstract. We establish existence, uniqueness and stability of transonic shocks for steady compressible non-isentropic potential flow system in a multidimensional divergent nozzle with an arbitrary smooth cross-section, for a prescribed exit pressure. The proof is based on solving a free boundary problem for a system of partial differential equations consisting of an elliptic equation and a transport equation. In the process, we obtain unique solvability for a class of transport equations with velocity fields of weak regularity(non-Lipschitz), an infinite dimensional weak implicit mapping theorem which does not require continuous Fréchet differentiability, and regularity theory for a class of elliptic partial differential equations with discontinuous oblique boundary conditions.

1. Introduction

We consider inviscid compressible steady flow of ideal polytropic gas.

One of important but difficult subjects in the study of transonic flow is to understand a global feature of flow through a convergent-divergent nozzle so called a *de Laval nozzle*. According to the quasi-linear approximation in [CFR, Chapter V. section 147], given incoming subsonic flow at the entrance, the flow is always accelerated through the convergent part of the nozzle unless the exit pressure exceeds the pressure at the entrance. The flow pattern through the divergent part, however, varies depending on the exit pressure and the shape of the nozzle. In the divergent part, the flow may remain subsonic all the way to the exit, or it may be accelerated to a supersonic state after passing the throat of the nozzle and have a transonic shock across which the velocity jumps down from supersonic to subsonic. In particular, the approximation implies that if the exit pressure is given by a constant p_c satisfying $0 < p_{min} < p_c < p_{max} < \infty$ for some constants p_{min} and p_{max}, then a transonic shock occurs in the divergent part of the nozzle. One may refer [CFR] for the quasi-linear approximation in curved nozzles. In section 2.4, for any given constant exit pressure $p_c \in (p_{min}, p_{max})$, we rigorously compute the transonic flow of the Euler system in a multidimensional nozzle expanding cone-shaped.

The issues above motivated recent works by several authors on existence and stability of steady transonic shocks in nozzles using the models of potential flow or compressible Euler system [CF1, CF2, CF3, XY1, CH1, YU1, CCF, XY2]. In these works transonic shocks were studied in cylindrical nozzles and its perturbations. A physically natural setup for transonic shock problem is to prescribe the parameters of flow on the nozzle entrance and the pressure on the nozzle exit. In particular, an important issue is to show that the shock location is uniquely determined by these parameters. However, such problem is not well posed in the cylindrical nozzles. Indeed, the flat shock between uniform states can be translated along the cylindrical nozzle, and this transform does not change the flow parameters on the nozzle entrance and the pressure on the nozzle exit. This provides an explicit example of non-uniqueness. Also, for uniform states in cylindrical nozzles, the flow on the nozzle entrance determines the pressure on the nozzle exit for a transonic shock solution. This degeneracy leads to non-existence, i.e.

one cannot prescribe an arbitrary (even if small) perturbation of pressure on the nozzle exit. On the other hand, as the de Laval nozzle example suggests, it is natural to study transonic shocks in the diverging nozzles. This removes the translation invariance in the case of Euler system. Recent works by Liu, Yuan[LY], S.-X. Chen [CH3], and by Li, Xin, Yin [LXY] showed well-posedness of transonic shock problem with prescribed flow on entrance and pressure on exit of a divergent nozzle, which are small perturbation of the corresponding parameters of the background solution, in the case of compressible Euler system in dimension two, under some additional restrictions on the perturbed pressure on the exit. In particular, it was assumed that for the prescribed pressure on exit, the normal derivatives on the nozzle walls vanish. While this condition is not physically natural, mathematically it leads to substantial simplifications: it allows to show C^2 regularity of the shock up to the nozzle walls, which allows to work with C^1 vector fields in the transport equations.

In this paper we study transonic shocks in diverging nozzles in any dimension, for the case of perturbed cone-shaped nozzle of arbitrary cross-section. Moreover, we do not make assumptions on the pressure on the nozzle exit, other than the smallness of the perturbation. This requires to consider shocks with regularity deteriorating near the nozzle walls, and then vector fields in transport equations are of low regularity (non-Lipschitz).

Furthermore, we study this problem in the framework of potential flow. A surprising feature of this problem is that it is not well-posed for the "standard" potential flow equation. Namely, for radial solutions in a cone-shaped nozzle, the spherical transonic shock can be placed in any location between the nozzle entrance and exit, without changing the parameters of flow on the nozzle entrance and exit. This follows from a calculation by G.-Q Chen, M. Feldman [CF1, pp. 488-489], see more details below. Thus, spherical shocks separating radial flows in the cone-shaped divergent nozzles can be translated along the nozzle for potential flow equation. This is similar to the case of flat shocks separating uniform flows in straight nozzles.

In order to remove this degeneracy, we introduce a new potential flow model, which we call *non-isentropic potential flow system*. It is based on the full Euler system (as opposed to the "standard" potential flow equation, which is based on the isentropic Euler system), and allows entropy jumps across the shock. The system consists of an equation of second order, which is elliptic in the subsonic regions and hyperbolic in the supersonic regions, and two transport equations.

The main purpose of this paper is to establish the existence, uniqueness and stability of a transonic shock for non-isentropic potential flow system in a multidimensional divergent nozzle with an arbitrary smooth cross-section provided that we smoothly perturb the cone-shaped divergent nozzle, incoming radial supersonic flow and the constant pressure at the exit.

In Section 2, we introduce the non-isentropic potential flow model, and define a transonic shock solution of this new model. Then we state two main theorems of this paper and present a framework of the proof for the theorems. Section 3 is devoted to a free boundary problem for the velocity potential φ where the exit normal velocity is fixed . In section 4, we plug a solution φ of the free boundary problem in section 3 to a transport equation to find the corresponding pressure p. In section 5, we prove the existence of a transonic shock solution of the non-isentropic potential flow for a fixed exit pressure p_{ex}. The uniqueness of the transonic shock solution is proven in section 6.

2. Main theorems

2.1. **Compressible Euler system and shocks.** We study transonic shocks of steady inviscid compressible flow of ideal polytropic gas. Such flow is governed by the conservation of mass, momentum and the conservation of energy. These three conservation laws provide a system of PDEs for the density ρ, velocity $\vec{u} = (u_1, \cdots, u_n)$ and the pressure p as follows:

$$\begin{aligned}
&div(\rho\vec{u}) = 0\\
&div(\rho\vec{u}\otimes\vec{u} + pI_n) = 0\\
&div\big(\rho\vec{u}(\frac{1}{2}|\vec{u}|^2 + \frac{\gamma p}{(\gamma-1)\rho})\big) = 0
\end{aligned}\tag{2.1}$$

where I_n is $n\times n$ identity matrix. The quantity c, given by

$$c := \sqrt{\frac{\gamma p}{\rho}},\tag{2.2}$$

is called the *sound speed*. The flow type can be classified by the ratio, *called the Mach number*, of $|\vec{u}|$ to c. If $\frac{|\vec{u}|}{c} > 1$ then the flow is said *supersonic*, if $\frac{|\vec{u}|}{c} < 1$ then the flow is said *subsonic*. If $\frac{|\vec{u}|}{c} = 1$ then the flow is said *sonic*. Depending on entrance and exit data of flow or a shape of a channel, there may be a jump transition of the flow type across a curve or a surface. Such a transition can be understood in a weak sense as follows.

For an open set $\Omega\subset\mathbb{R}^n$, if $(\rho,\vec{u},p)\in L^1_{loc}(\Omega)$ satisfies

$$\int_\Omega \rho\vec{u}\cdot\nabla\xi dx = \int_\Omega (\rho u_k\vec{u} + p\hat{e}_k)\cdot\nabla\xi dx = \int_\Omega \rho\vec{u}\big(\frac{1}{2}|\vec{u}|^2 + \frac{\gamma p}{(\gamma-1)}\big)\cdot\nabla\xi dx = 0\tag{2.3}$$

for any $\xi\in C_0^\infty(\Omega)$ and all $k = 1,\cdots,n$ where $\hat{e}_k$ is a unit normal in k-th direction for $\mathbb{R}^n$, $(\rho,\vec{u},p)$ is called a *weak solution* to (2.1) in Ω.

Suppose that a smooth (n-1)-dimensional surface S divides Ω into two disjoint subsets $\Omega^\pm$. If $(\rho,\vec{u},p)\in L^1_{loc}(\Omega)$ is in $C^1(\Omega^\pm)\cap C^0(\overline{\Omega^\pm})$ then $(\rho,\vec{u},p)$ is a weak solution of (2.1) if and only if $(\rho,\vec{u},p)$ satisfies (2.1) in each of $\Omega^\pm$ and the *Rankine-Hugoniot jump conditions*(which is abbreviated as *R-H conditions* hereafter)

$$[\rho\vec{u}\cdot\nu_s]_S = 0\tag{2.4}$$

$$[\rho(\vec{u}\cdot\nu_s)\vec{u} + p\nu_s]_S = \vec{0}\tag{2.5}$$

$$[\frac{1}{2}|\vec{u}|^2 + \frac{\gamma p}{(\gamma-1)\rho}]_S = 0\tag{2.6}$$

for a unit normal ν_s on S where $[F]_S$ is defined by

$$[F(x)]_S := F(x)|_{\overline{\Omega^-}} - F(x)|_{\overline{\Omega^+}} \quad \text{for } x\in S.$$

To study a transonic shock in $\mathbb{R}^n$ for $n\geq 2$, we consider an irrotational flow in which the velocity $\vec{u}$ has an expression of $\vec{u} = \nabla\varphi$ for a scalar function φ. Such a flow is called a *potential flow* and φ is called a *(velocity) potential*.

2.2. **Steady isentropic potential flow equation and radial transonic shocks.** A widely used steady potential flow model(see e.g. [CF1, XY1] et al.) consists of the conservation law of

mass, the Bernoulli law for the velocity. So it can be written into the following second-order, nonlinear elliptic-hyperbolic equation of mixed type for the velocity potential $\varphi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^1$:

$$\mathrm{div}(\rho(|\nabla\varphi|^2)\nabla\varphi) = 0, \tag{2.7}$$

where, with a particular choice of the Bernoulli's invariant, the density function $\rho(q^2)$ is

$$\rho(q^2) = \left(1 - \frac{\gamma-1}{2}q^2\right)^{\frac{1}{\gamma-1}}. \tag{2.8}$$

We note that equation (2.7) can be derived formally from the isentropic Euler system. Steady isentropic Euler system is obtained from (2.1) by setting $p = \kappa\rho^\gamma$, where $\kappa > 0$ is a constant, and dropping the third line in (2.1). Then equation (2.7) can be obtained by a formal calculation, substituting $\vec{u} = \nabla\varphi$ into the isentropic Euler system, and choosing $\kappa = \frac{1}{\gamma}$.

$$\begin{aligned} &\textit{The second-order nonlinear equation (2.7) is} \\ &\textit{strictly elliptic if } |\nabla\varphi| > c, \textit{ and is strictly hyperbolic if } |\nabla\varphi| < c \end{aligned} \tag{2.9}$$

with $c = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\rho^{\gamma-1}}$. The elliptic regions of the equation (2.7) correspond to the subsonic flow, and the hyperbolic regions of (2.7) to the supersonic flow.

From the divergence structure of (2.7), it is easy to derive the following RH conditions on shocks. Suppose that a smooth (n-1)-dimensional surface S divides Ω into two disjoint subsets $\Omega^\pm$. If $\varphi \in C^{0,1}(\Omega)$ is in $C^2(\Omega^\pm) \cap C^1(\Omega^\pm \cup S)$ then φ is a a weak solution of (2.7) if and only if φ satisfies (2.7) in each of $\Omega^\pm$ and

$$\left[\rho(|\nabla\varphi|^2)\nabla\varphi \cdot \nu\right]_S = 0. \tag{2.10}$$

Note that $[\varphi]_S = 0$ since $\varphi \in C^{0,1}(\Omega)$.

Shock S is called transonic if it separates subsonic and supersonic regions.

Now we discuss radial transonic shocks, i.e. transonic shock solutions of the form

$$\varphi(x) = w(r) \quad \text{where} \quad r = |x|.$$

Let $0 < r_1 < r_2$

$$\Omega = \{r_1 < |x| < r_2\}.$$

Then calculation in [CF1, pp. 488-489] show that there exists (radial) functions $\varphi_0^\pm(x) = w_0^\pm(|x|$ (defined by [CF1, eqn (7.4)] with e.g. $R_0 := r_1$) such that

(i) $\varphi_0^\pm \in C^\infty(\Omega)$ and satisfy (2.7) in Ω,

(ii) φ_0^- is supersonic and φ_0^+ is subsonic in Ω. Moreover $0 < \max_{r\in[r_1,r_2]} \frac{dw_0^+}{dr}(r) < \min_{r\in[r_1,r_2]} \frac{dw_0^-}{dr}(r)$.

(iii) for any $r_s \in (r_1, r_2)$, choosing a constant $c = c(r_s)$ so that $w_0^+(r_s) = w_0^-(r_s) + c$, there holds: function

$$\varphi_0(x) = \min(\varphi_0^+(x), \varphi_0^-(x) + c(r_s)) \quad \text{for} \quad x \in \Omega$$

is a transonic shock solution of (2.7) with shock $S = \{|x| = r_s\}$, and $\varphi_0 = \varphi_0^-$ in $\Omega_{r_s}^- := \{r_1 < |x| < r_s\}$; and $\varphi_0 = \varphi_0^+$ in $\Omega_{r_s}^+ := \{r_s < |x| < r_2\}$. In particular φ_0 is supersonic in $\Omega_{r_s}^-$ and subsonic in $\Omega_{r_s}^+$.

Note that $\partial_r\varphi_0^{\pm} > 0$. Thus flow enters Ω through $\{|x| = r_1\}$ and exits Ω through $\{|x| = r_2\}$. By part(iii) above, $\nabla\varphi$ does not change on the entrance and exit when the position r_s of the shock changes within (r_1, r_2). Since $\vec{u} = \nabla\varphi$ determines the density ρ by (2.8) and pressure $p = \kappa\rho^{\gamma}$, it follows that parameters of the flow on the entrance and exist of the annulus-shaped domain Ω do not determine position $r = r_s$ of the shock.

Restricting the annulus-shaped domain Ω to a cone-shaped (nozzle) domain $\mathcal{N}$ as in Section 2.4, we see that the the same example shows non-uniqueness of the shock position in the cone-shaped nozzle.

Thus we propose a new potential model, to fix this degeneracy.

2.3. **The non-isentropic potential flow system and transonic shocks.** The new model, which we call *the non-isentropic potential flow system*, is obtained by substituting $\vec{u} = \nabla\varphi$ into the the full steady compressible Euler system (2.1). Unknown functions in the new model are the scalar functions (φ, ρ, p). Note that second line of (2.1), representing conservation of momentum, consists of n equations. Thus we have overdeterminacy in the case of potential model, i.e. we need to retain one equation from the conservation of momentum equations. For that, we note that the component in the direction of $\vec{u}$ of the vector determined by the second line of (2.1) represents conservation of entropy along streamlines (in the smooth flow regions). This motivates us to introduce the following non-isentropic potential flow system:

$$div(\rho\nabla\varphi) = 0, \tag{2.11}$$

$$\nabla\varphi \cdot div(\rho\nabla\varphi \otimes \nabla\varphi + pI_n) = 0, \tag{2.12}$$

$$div\big(\rho\nabla\varphi(\frac{1}{2}|\nabla\varphi|^2 + \frac{\gamma p}{(\gamma-1)\rho})\big) = 0. \tag{2.13}$$

(2.11) and (2.13) represent the conservation of mass and energy respectively, and (2.12) concerns the conservation of the entropy along each streamline but it allows the entropy to change between streamlines.

Since (2.12) is not in a divergence form, we cannot define a shock solution for this new model in sense of (2.3). Instead, we employ all the R-H conditions (2.4)-(2.6) to define a shock solution of the new model as follows:

Definition 1 (Shock solutions). (ρ, φ, p) *is a shock solution of the non-isentropic potential flow in* Ω *with a shock* S *if*

(i) $\rho, p \in C^0(\overline{\Omega^{\pm}}) \cap C^1(\Omega^{\pm})$, $\varphi \in C^1(\overline{\Omega^{\pm}}) \cap C^2(\Omega^{\pm})$;

(ii) $(\rho, \nabla\varphi, p)|_{S\cap\overline{\Omega^-}} - (\rho, \nabla\varphi, p)|_{S\cap\overline{\Omega^+}} \neq \vec{0}$;

(iii) (ρ, φ, p) *satisfies* (2.11)-(2.13) *pointwise in* $\Omega^{\pm}$, *and* (2.4)-(2.6) *on* S *with* $\vec{u} = \nabla\varphi$.

For $(\rho, \varphi, p) \in C^{\infty}$, if $\rho > 0$ then (2.11)-(2.13) are equivalent to

$$div\big((B - \frac{1}{2}|\nabla\varphi|^2)^{\frac{1}{\gamma-1}}\nabla\varphi\big) = 0 \tag{2.14}$$

$$\nabla\varphi \cdot \nabla\frac{p}{(B - \frac{1}{2}|\nabla\varphi|^2)^{\frac{\gamma}{\gamma-1}}} = 0 \tag{2.15}$$

$$\nabla\varphi \cdot \nabla B = 0 \tag{2.16}$$

where the *Bernoulli's invariant* B is defined by

$$B = \frac{1}{2}|\nabla\varphi|^2 + \frac{\gamma p}{(\gamma-1)\rho}. \tag{2.17}$$

An explicit computation shows that, as a second-order PDE for φ, (2.14) with (2.17) is

$$\text{strictly elliptic if } |\nabla\varphi| < c, \text{ and is strictly hyperbolic if } |\nabla\varphi| > c$$

with c defined in (2.2). Note that this is similar to (2.9).

Definition 2 (A transonic shock solution)**.** *A shock solution (ρ, φ, p), defined in Definition 1, is a transonic shock solution if (i)$\partial_{\nu_s}\varphi|_{\overline{\Omega^-}\cap S} > \partial_{\nu_s}\varphi|_{\overline{\Omega^+}\cap S} > 0$ for the unit normal ν_s on S pointing toward Ω^+, and (ii) $|\nabla\varphi| > c$ in Ω^-, $|\nabla\varphi| < c$ in Ω^+ where c is defined in* (2.2)*.*

Let τ_s be a tangent vector field on S, then, by (2.5), there holds

$$[\rho(\nabla\varphi\cdot\nu_s)(\nabla\varphi\cdot\tau_s)]_S = 0. \tag{2.18}$$

Assuming no vacuum state in Ω i.e., $\rho \neq 0$ in Ω, (2.4) and (2.18) imply either $\nabla\varphi\cdot\nu_s = 0$ or $[\nabla\varphi\cdot\tau_s]_S = 0$. If $\nabla\varphi\cdot\nu_s = 0$ then Definition 3.34 (ii) implies $[\nabla\varphi\cdot\tau_s]_S \neq 0$. In that case, S becomes a vortex sheet. If $[\nabla\varphi\cdot\tau_s]_S = 0$ holds on S then this implies $[\varphi]_S = k$ for some constant k. We are interested in the case of $\nabla\varphi\cdot\nu_s \neq 0$ on S so (2.18) provides

$$[\varphi]_S = 0 \tag{2.19}$$

choosing the constant $k = 0$ without loss of generality. In addition, $(2.5)\cdot\nu_s$ provides

$$[\rho(\nabla\varphi\cdot\nu_s)^2 + p]_S = 0. \tag{2.20}$$

Remark 2.1. *The specific entropy ϵ of ideal polytropic gas satisfies the constitutive relation*

$$\kappa\exp(\epsilon/c_v) = \frac{p}{\rho^\gamma}$$

for constants κ and c_v. A straightforward calculation using (2.4),(2.6),(2.19) *and* (2.20) *indicates that the entropy ϵ increases across a shock S for the non-isentropic potential flow model. The entropy jump across a shock plays an essential role for the well-posedness of the problem considered in this paper.*

2.4. **Radial transonic shock solutions.** Fix an open connected set $\Lambda \subset \mathbb{S}^{n-1}(n \geq 2)$ with a smooth boundary $\partial\Lambda$, and define a straight divergent nozzle $\mathcal{N}$ by

$$\mathcal{N} := \{x : r_0 < |x| < r_1, \frac{x}{|x|} \in \Lambda\} \quad \text{for } 0 < r_0 < r_1 < \infty.$$

Regarding Λ as a n-1 dimensional submanifold of $\mathbb{R}^{n-1}$, let $x' = (x'_1, \cdots, x'_{n-1})$ be a coordinate system in Λ. Particularly, if $\Lambda \subsetneq \mathbb{S}^{n-1}$, then Λ can be described by a single smooth diffeomorphism(e.g. stereographic projection), say, T i.e., there is $\Lambda_T \subset \mathbb{R}^{n-1}$ with a smooth boundary so that Λ_T can be described by

$$\Lambda = \{T(x') : x' \in \Lambda_T\}. \tag{2.21}$$

Then, for any $x \in \mathcal{N}$, there exists unique x' satisfying $\frac{x}{|x|} = T(x')$. For convenience, we denote as

$$x =_T (r, x') \quad x' \in_T \Lambda \quad \text{if } r = |x|, \quad T(x') = \frac{x}{|x|} \in \Lambda. \tag{2.22}$$

To simplify notations, we denote as $x' \in \Lambda$ instead of $x' \in_T \Lambda$ hereafter. The coordinates (r, x') in (2.22) is regarded as a spherical coordinate system in $\mathcal{N}$.

For later use, we set

$$\Gamma_{ent} := \{x =_T (r, x') : r = r_0, x' \in \Lambda\}, \quad \Gamma_{ex} := \{x =_T (r, x') : r = r_1, x' \in \Lambda\},$$
$$\Gamma_w := \partial\mathcal{N} \setminus (\Gamma_{ent} \cup \Gamma_{ex}).$$

To simplify notations, for a function $\psi(x)$ defined for $x \in \mathcal{N}$, if $x =_T (r, x')$ then we identify $\psi(r, x')$ with $\psi(x)$ without any further specification for the rest of paper. Similarly, any function $f(\frac{x}{|x|})$ defined for $\frac{x}{|x|} \in \Lambda$, we identify $f(x')$ with $f(\frac{x}{|x|})$ if $\frac{x}{|x|} = T(x')$. Furthermore, any function φ_0 varying only by the radial variable $|x|$, we write as $\varphi_0(r)$. Also, in (r, x') coordinates, we denote as

$$\mathcal{N} = (r_0, r_1) \times \Lambda_T, \;\; \mathcal{N}_a^- = \mathcal{N} \cap \{r < a\}, \;\; \mathcal{N}_a^+ = \mathcal{N} \cap \{r > a\}. \tag{2.23}$$

For a fixed constant vector $(\rho_{in}, v_{in}, p_{in}) \in \mathbb{R}^3_+$ with $v_{in} > c_{in}(= \sqrt{\frac{\gamma p_{in}}{\rho_{in}}})$, let us find a transonic shock solution (ρ_0, φ_0, p_0) satisfying

$$(\rho_0, \partial_r \varphi_0, p_0) = (\rho_{in}, v_{in}, p_{in}) \quad \text{on } \Gamma_{ent}, \tag{2.24}$$

$$\partial_{\nu_w} \varphi_0 = 0 \quad \text{on } \Gamma_w \tag{2.25}$$

where ν_w is the unit normal on Γ_w toward the interior of $\mathcal{N}$. Since (2.11)-(2.13) are invariant under rotations, we expect (ρ_0, φ_0, p_0) to be functions of r only. Then (2.25) automatically holds on Γ_w. For a continuously differentiable radial solution (ρ, φ, p), if $\varphi_r \neq 0$ in $\mathcal{N}$, then (2.11)-(2.13) are equivalent to

$$\begin{aligned} &\frac{d}{dr}(r^{n-1}\rho\varphi_r) = 0, \\ &\rho\frac{d}{dr}(\frac{1}{2}\varphi_r^2) + \frac{dp}{dr} = 0, \\ &\frac{1}{2}\varphi_r^2 + \frac{\gamma p}{(\gamma-1)\rho} = \frac{1}{2}v_{in}^2 + \frac{\gamma p_{in}}{(\gamma-1)\rho_{in}} =: B_0. \end{aligned} \tag{2.26}$$

(2.26) can be considered as a system of ODEs for (φ_r, ρ) because of

$$\rho = \frac{\gamma p}{(\gamma-1)(B_0 - \frac{1}{2}\varphi_r^2)}. \tag{2.27}$$

To find (φ_r, p), we rewrite (2.26) as

$$\frac{d\varphi_r}{dr} = \frac{2(n-1)(\gamma-1)\varphi_r(B_0 - \frac{1}{2}\varphi_r^2)}{(\gamma+1)r(\varphi_r^2 - K_0)}, \tag{2.28}$$

$$\frac{dp}{dr} = -\frac{2(n-1)\gamma\varphi_r^2 p}{(\gamma+1)r(\varphi_r^2 - K_0)} \tag{2.29}$$

with

$$K_0 := \frac{2(\gamma-1)B_0}{\gamma+1}. \tag{2.30}$$

Fix $r_s \in (r_0, r_1)$, and let us find a radial transonic shock solution with the shock on $\{r = r_s\}$. For that purpose, we need to solve (2.28) and (2.29) in two separate regions of supersonic state and subsonic state.

For the supersonic region $\mathcal{N}_{r_s}^-$, the initial condition is

$$(\varphi_r, p)(r_0) = (v_{in}, p_{in}). \tag{2.31}$$

Then (2.28), (2.29) with (2.31) have a unique solution $(\partial_r\varphi_0^-, p_0^-)$ in $\mathcal{N}_{r_s}^-$. For the subsonic region $\mathcal{N}_{r_s}^+$, we solve the system of the algebraic equations (2.4), (2.20) and (2.27) with $\partial_r\varphi_0^-$

and p_0^- on the side of $\mathcal{N}_{r_s}^-$ for $(\partial_r\varphi, p)(r_s)$. Then there are two solutions for $(\partial_r\varphi, p)(r_s)$ but the only admissible solution in sense of Definition 2 is

$$\partial_r\varphi(r_s) = \frac{K_0}{\partial_r\varphi_0^-(r_s)}, \tag{2.32}$$

$$p(r_s) = (\rho_0^-(\partial_r\varphi_0^-)^2 + p_0^- - \rho_0^- K_0)(r_s) =: p_{s,0}(r_s). \tag{2.33}$$

So the initial condition for the subsonic region is given by (2.32) and (2.33).

Let $(\partial_r\varphi_0^+, p_0^+)$ be the solution to (2.28), (2.29) with (2.32), (2.33).

We claim that $(\partial_r\varphi_0^-, p_0^-)$ is indeed supersonic in $\mathcal{N}_{r_s}^-$ while $(\partial_r\varphi_0^+, p_0^+)$ is subsonic in $\mathcal{N}_{r_s}^+$. Plug $\frac{\gamma p}{\rho} = c^2$ into the equation of Bernoulli's law $\frac{1}{2}(\partial_r\varphi)^2 + \frac{\gamma p}{(\gamma-1)\rho} = B_0$ to obtain

$$\varphi_r^2 - c^2 = \frac{\gamma+1}{2}(\varphi_r^2 - K_0). \tag{2.34}$$

Denote $\sqrt{\frac{\gamma p_0{}^{\pm}}{\rho_0^{\pm}}}$ as $c_{\pm}$ then, $v_{in}^2 > c_{in}^2$ and (2.34) imply $\frac{d\varphi_r}{dr} > 0$ for $\varphi_r = \partial_r\varphi_0^-$ thus $(\partial_r\varphi_0^-)^2 > c_-^2$ for $r \geq r_0$ so $(\partial_r\varphi_0^-, p_0^-)$ is supersonic in $\mathcal{N}_{r_s}^-$. On the other hand, by (2.32), $\partial_r\varphi_0^+(r_s)$ satisfies the inequality

$$(\partial_r\varphi_0^+)^2 - c_+^2 = \frac{\gamma+1}{2}\frac{K_0}{(\partial_r\varphi_0^-)^2}\big(K_0 - (\partial_r\varphi_0^-)^2\big) < 0 \tag{2.35}$$

at $r = r_s$, and this implies $\frac{d\varphi_r}{dr} < 0$ for $\varphi_r = \partial_r\varphi_0^+$. So $(\partial_r\varphi_0^+)^2 < c_+^2$ holds for $r \geq r_s$. Therefore, we get a family of radial transonic shock solutions as follows:

Definition 3 (**Background solutions**). *For $r_s \in (r_0, r_1)$, define*

$$(\rho_0, \varphi_0, p_0)(r; r_s) := \begin{cases} (\rho_0^-, \varphi_0^-, p_0^-)(r; r_s) & \text{in } [r_0, r_s] \\ (\rho_0^+, \varphi_0^+, p_0^+)(r; r_s) & \text{in } (r_s, r_1] \end{cases}$$

with $\rho_0^{\pm}(r; r_s) := \frac{\gamma p_0^{\pm}(r)}{(\gamma-1)(B_0 - \frac{1}{2}(\partial_r\varphi_0^{\pm}(r))^2)}$, and

$$\varphi_0^-(r; r_s) := \int_{r_0}^r \partial_r\varphi_0^-(t)dt, \quad \varphi_0^+(r; r_s) = \int_{r_s}^r \partial_r\varphi_0^+(t)dt + \varphi_0^-(r_s)$$

where $(\partial_r\varphi_0^{\pm}, p_0^{\pm})(r)$ is the solution to (2.28)-(2.29) with (2.32)-(2.33) and (2.31) respectively. We call $(\rho_0, \varphi_0, p_0)(r; r_s)$ the background solution with the transonic shock on $\{r = r_s\}$.

Remark 2.2. *For a small constant $\delta > 0$, $(\partial_r\varphi_0^{\pm}, p_0^{\pm})$ can be extended to $\mathcal{N}_{r_s-2\delta}^+$ and $\mathcal{N}_{r_s+2\delta}^-$ as solutions to (2.28)-(2.29) with (2.32)-(2.33) and (2.31) respectively where $\mathcal{N}_{r_s\mp\delta}^{\pm}$ is defined by (2.23).*

Remark 2.3. *For a fixed r_s, ρ_0^- and p_0^- monotonically decrease while $\partial_r\varphi_0^-$ monotonically increases by r. On the other hand, by (2.35), ρ_0^+ and p_0^+ monotonically increase while $\partial_r\varphi_0^+$ monotonically decreases.*

Another important property of the background solutions is the monotonicity of the exit values of $(\rho_0, \partial_r\varphi_0, p_0)$ depending on the location of shocks r_s. In [YU2], it is proven that, for the full Euler system, $\rho_0^+(r_1; r_s)$ and $p_0^+(r_1; r_s)$ monotonically decrease with respect to r_s while $\partial_r\varphi_0^+(r_1; r_s)$ monotonically increases. If $\partial_r\varphi_0(r; r_s) \neq 0$ in $\mathcal{N}$, then the non-isentropic potential flow model is equivalent to the full Euler system for smooth flow. So the monotonicity

properties of the exit data applies to the background solution (ρ_0, φ_0, p_0) of the non-isentropic potential flow model so we have:

Proposition 2.4 ([YU2, Theorem 2.1])**.** *Let us set $p_{min} = p_0^+(r_1; r_1)$ and $p_{max} := p_0^+(r_1; r_0)$. Then $p_{min} < p_{max}$ holds. Moreover, for any constant $p_c \in (p_{min}, p_{max})$, there exists unique $r_s^* \in (r_0, r_1)$ so that the background solution $(\rho_0, \varphi_0, p_0)(r; r_s^*)$ satisfies $p_0^+(r_1; r_s^*) = p_c$.*

Readers can refer [YU2] for the proof although we would like to point out that the following lemma, which will be used later in this paper, plays an important role for the proof of Proposition 2.4.

Lemma 2.5. *For any given $r_s \in (r_0, r_1)$, let $(\rho_0^+, \varphi_0^+, p_0^+)(r)$ be the background transonic shock solution with the shock on $\{r = r_s\}$ defined in Definition 3, and let us define μ_0 by*

$$\mu_0 := \frac{\frac{d}{dr}(\frac{K_0}{\partial_r \varphi_0^-} - \partial_r \varphi_0^+)(r_s)}{\partial_r(\varphi_0^- - \varphi_0^+)(r_s)}.$$

(a) *Then, there holds $\mu_0 > 0$.*
(b) *Also, we have, for $p_{s,0}$ defined in (2.33)*

$$[\frac{dp_{s,0}}{dr} - \frac{dp_0^+}{dr}](r_s) = -\frac{(n-1)\gamma p_0^- [(\partial_r \varphi_0^-)^2 + \frac{\gamma-1}{\gamma+1} K_0]}{(\gamma-1) r (B_0 - \frac{1}{2}(\partial_r \varphi_0^-)^2)}|_{r=r_s} < 0. \tag{2.36}$$

Proof. By (2.28), we have

$$\begin{aligned} &\frac{d}{dr}(\frac{K_0}{\partial_r \varphi_0^-} - \partial_r \varphi_0^+)(r_s) \\ &= -\frac{C(n,\gamma)}{r_s}[\frac{K_0(B_0 - \frac{1}{2}(\partial_r \varphi_0^-)^2)}{\partial_r \varphi_0^-((\partial_r \varphi_0^-)^2 - K_0)} + \frac{\partial_r \varphi_0^+(B_0 - \frac{1}{2}(\partial_r \varphi_0^+)^2)}{(\partial_r \varphi_0^+)^2 - K_0}](r_s). \end{aligned}$$

with the constant $C(n,\gamma) = \frac{2(n-1)(\gamma-1)}{\gamma+1} > 0$. Applying (2.32) to replace $\partial_r \varphi_0^+(r_s)$ by $\frac{K_0}{\partial_r \varphi_0^-(r_s)}$, we obtain

$$\frac{d}{dr}(\frac{K_0}{\partial_r \varphi_0^-} - \partial_r \varphi_0^+)(r_s) = \frac{2(n-1)\gamma K_0}{(\gamma+1) r_s \partial_r \varphi_0^-(r_s)} > 0. \tag{2.37}$$

By (2.32) and (2.35), it is easy to see $\partial_r(\varphi_0^- - \varphi_0^+)(r_s) > 0$, and thus (a) holds true for all $r_s \in (r_0, r_1)$.

Similarly, by (2.27)-(2.29), (2.32) and (2.33), we obtain (2.36). □

2.5. **Main theorems.** According to Proposition 2.4, for any given constant $p_c \in (p_{min}, p_{max})$, there exists a transonic shock solution whose exit pressure is p_c. Our goal is to achieve the existence of a transonic shock solution when we smoothly perturb incoming supersonic flow, the exit pressure and the nozzle $\mathcal{N}$.

Let $\widetilde{\mathcal{N}}$ be a nozzle smoothly perturbed from $\mathcal{N}$ by a diffeomorphism $\Psi : \mathcal{N} \to \mathbb{R}^n$, and $(\tilde\rho, \tilde\varphi, \tilde p)$ be a transonic shock solution of a non-isentropic potential flow in $\widetilde{\mathcal{N}}$ with a shock $\widetilde{S}$. Then $\widetilde{S}$ separates $\widetilde{\mathcal{N}}$ into a supersonic region $\widetilde{\mathcal{N}}^-$ and a subsonic region $\widetilde{\mathcal{N}}^+$. Let us denote $(\tilde\rho, \tilde\varphi, \tilde p)|_{\widetilde{\mathcal{N}}^\pm}$ as $(\tilde\rho_\pm, \tilde\varphi_\pm, \tilde p_\pm)$. If the entrance data for $(\tilde\rho, \tilde\varphi, \tilde p)$ is given so that the Bernoulli's invariant B is a constant at the entrance of the nozzle $\widetilde{\mathcal{N}}$ then, by (2.4), (2.6) and (2.16),

$(\tilde{\rho}_\pm, \tilde{\varphi}_\pm, \tilde{p}_\pm)$ is a solution to the system of (2.14), (2.15) and

$$\frac{1}{2}|\nabla\varphi|^2 + \frac{\gamma p}{(\gamma-1)\rho} = B_0 \quad \text{in} \quad \widetilde{\mathcal{N}}^\pm \tag{2.38}$$

where the constant $B_0 > 0$ is determined by the entrance data.

On the shock $\widetilde{S}$, (2.19) and (2.38) imply

$$\Big[\frac{1}{2}(\nabla\tilde{\varphi}\cdot\tilde{\nu}_s)^2 + \frac{\gamma\tilde{p}}{(\gamma-1)\tilde{\rho}}\Big]_{\widetilde{S}} = 0 \tag{2.39}$$

for a unit normal $\tilde{\nu}_s$ on $\widetilde{S}$.

Let us set $\widetilde{K}_s := \frac{2(\gamma-1)}{\gamma+1}\big(\frac{1}{2}(\nabla\tilde{\varphi}_-\cdot\tilde{\nu}_s)^2 + \frac{\gamma\tilde{p}_-}{(\gamma-1)\tilde{\rho}_-}\big)$, and solve (2.4), (2.20) and (2.39) for $\nabla\tilde{\varphi}_+\cdot\tilde{\nu}_s$ and $\tilde{p}_+$ to obtain

$$\nabla\tilde{\varphi}_+\cdot\tilde{\nu}_s = \frac{\widetilde{K}_s}{\nabla\tilde{\varphi}_-\cdot\tilde{\nu}_s}, \tag{2.40}$$

$$\tilde{p}_+ = \tilde{\rho}_-(\nabla\tilde{\varphi}_-\cdot\tilde{\nu}_s)^2 + \tilde{p}_- - \tilde{\rho}_-\widetilde{K}_s \quad \text{on} \quad \widetilde{S}. \tag{2.41}$$

For a transonic shock solution $(\tilde{\rho}, \tilde{\varphi}, \tilde{p})$ in $\widetilde{\mathcal{N}}$, let $\widetilde{\mathcal{N}}^- := \{|\nabla\tilde{\varphi}| > c\}$ be the supersonic region, and $\widetilde{\mathcal{N}}^+ := \{|\nabla\tilde{\varphi}| < c\}$ be the subsonic region where c is defined in (2.2) by replacing ρ, p by $\tilde{\rho}, \tilde{p}$ on the right-hand side.

Problem 1 (Transonic shock problem in a perturbed nozzle $\widetilde{\mathcal{N}} = \Psi(\mathcal{N})$)**.** *Let $(\tilde{\rho}_-, \tilde{\varphi}_-, \tilde{p}_-)$ be a supersonic solution upstream with $B = B_0$ at the entrance $\widetilde{\Gamma}_{ent} = \Psi(\Gamma_{ent})$ for a constant B_0, and suppose that $\nabla\tilde{\varphi}_-\cdot\nu_w = 0$ on $\widetilde{\Gamma}_w = \Psi(\Gamma_w)$ for a unit normal ν_w on $\widetilde{\Gamma}_w$.*

Given an exit pressure function $\tilde{p}_{ex}$ on $\widetilde{\Gamma}_{ex} = \Psi(\Gamma_{ex})$, locate a transonic shock $\widetilde{S}$ i.e., find a function f satisfying

$$\widetilde{\mathcal{N}}^- = \widetilde{\mathcal{N}} \cap \{|x| < f(\frac{x}{|x|})\}, \quad \widetilde{\mathcal{N}}^+ = \widetilde{\mathcal{N}} \cap \{|x| > f(\frac{x}{|x|})\},$$

and then find a corresponding subsonic solution $(\tilde{\rho}_+, \tilde{\varphi}_+, \tilde{p}_+)$ downstream so that

(i) *$(\tilde{\rho}_+, \tilde{\varphi}_+, \tilde{p}_+)$ satisfies (2.14), (2.15) and (2.38) in $\widetilde{\mathcal{N}}^+$;*
(ii) *$\tilde{\varphi}_+$ satisfies the slip boundary condition $\nabla\tilde{\varphi}_+\cdot\nu_w = 0$ on $\widetilde{\Gamma}_w^+ = \partial\widetilde{\mathcal{N}}^+ \cap \widetilde{\Gamma}_w$;*
(iii) *$(\tilde{\varphi}_+, \tilde{p}_+)$ satisfies (2.19), (2.40) and (2.41) on $\widetilde{S}$;*
(iv) *$\tilde{p}_+ = \tilde{p}_{ex}$ holds on $\widetilde{\Gamma}_{ex}$.*

For convenience, we reformulate Problem 1 for $(\rho, \varphi, p) = (\tilde{\rho}, \tilde{\varphi}, \tilde{p}) \circ \Psi$ in the unperturbed nozzle $\mathcal{N}$. If $(\tilde{\rho}, \tilde{\varphi}, \tilde{p})$ solves (2.14), (2.15), (2.38) then (ρ, φ, p) satisfies

$$div A(x, D\Psi, D\varphi) = 0 \tag{2.42}$$

$$(D\Psi^{-1})^T(D\Psi^{-1})D\varphi\cdot D\frac{p}{(B_0 - \frac{1}{2}|D\Psi^{-1}D\varphi|^2)^{\frac{\gamma}{\gamma-1}}} = 0 \tag{2.43}$$

$$\frac{1}{2}|D\Psi^{-1}D\varphi|^2 + \frac{\gamma p}{(\gamma-1)\rho} = B_0 \tag{2.44}$$

for $x \in \Psi^{-1}(\widetilde{\mathcal{N}}^\pm) =: \mathcal{N}^\pm$ with $D\varphi = (\partial_{x_j}\varphi)_{j=1}^n$, $D\Psi = (\partial_{x_i}\Psi_j)_{i,j=1}^n$, $D\Psi^{-1} = (D\Psi)^{-1}$. Here, $A(x, D\Psi, D\varphi)$ in (2.42) is defined by

$$A(x, m, \eta) = \det m\,(B_0 - \frac{1}{2}|m^{-1}\eta|^2)^{\frac{1}{\gamma-1}}(m^{-1})^T m^{-1}\eta \tag{2.45}$$

for $(x,m,\eta)\in\mathcal{N}\times B_d^{(n\times n)}(I_n)\times B_R^{(n)}(0)$ where we set $B_{\mathcal{R}}^{(k)}(a)$ as a closed ball of the radius $\mathcal{R}$ in $\mathbb{R}^k$ with the center at $a\in\mathbb{R}^k$. Here, d is a sufficiently small so that (2.45) is well defined for all $m\in B_d^{n\times n}(I_n)$.

By (2.19), the unit normal on $\widetilde{S}$ toward $\widetilde{\mathcal{N}}^+$ has the expression of

$$\tilde{\nu}_s=\frac{D_y(\tilde{\varphi}_- -\tilde{\varphi}_+)}{|D_y(\tilde{\varphi}_- -\tilde{\varphi}_+)|}\ \text{ with }\ D_y=(\partial_{y_1},\cdots,\partial_{y_n}). \tag{2.46}$$

So the R-H conditions for (ρ,φ,p) on $S:=\Psi^{-1}(\widetilde{S})$ are

$$\varphi_+=\varphi_- \tag{2.47}$$

$$Q^*(D\Psi,D\varphi_-,D\varphi_+)D\varphi_+\cdot\nu_s=\frac{K_s}{Q^*(D\Psi,D\varphi_-,D\varphi_+)D\varphi_-\cdot\nu_s} \tag{2.48}$$

$$p=\rho_-[(D\Psi^{-1})^T(D\Psi^{-1})D\varphi_-\cdot\nu_s]^2+p_- -\rho_-K_s \tag{2.49}$$

with $D=(\partial_{x_1},\cdots,\partial_{x_n})$ where we set

$$Q^*(m,\xi,\eta):=\frac{|\xi-\eta|}{|m^{-1}(\xi-\eta)|}(m^{-1})^T(m^{-1}), \tag{2.50}$$

$$K_s=\frac{2(\gamma-1)}{\gamma+1}\Big(\frac{1}{2}\Big(\frac{|D(\varphi_- -\varphi)|(D\Psi^{-1})^TD\Psi^{-1}D\varphi_-\cdot\nu_s}{|D\Psi^{-1}D(\varphi_- -\varphi)|}\Big)^2+\frac{\gamma p_-}{(\gamma-1)\rho_-}\Big), \tag{2.51}$$

$$\nu_s=\frac{D(\varphi_- -\varphi_+)}{|D(\varphi_- -\varphi_+)|}. \tag{2.52}$$

We note that, by (2.47), ν_s is the unit normal on S toward $\mathcal{N}^+$. Problem 1 is equivalent to:

Problem 2 (Transonic shock problem in the unperturbed nozzle $\mathcal{N}$)**.** *Let Ψ be a smooth diffeomorphism in $\mathcal{N}$. Given a supersonic solution $(\rho_-,\varphi_-,p_-)=(\tilde{\rho}_-,\tilde{\varphi}_-,\tilde{p}_-)\circ\Psi$ in $\mathcal{N}^-$ where $(\tilde{\rho}_-,\tilde{\varphi}_-,\tilde{p}_-)$ is as in Problem 1, and an exit pressure function $p_{ex}=\tilde{p}_{ex}\circ\Psi$, locate a transonic shock S i.e., find a function f satisfying $\mathcal{N}^-=\{|D\Psi^{-1}D\varphi|>c\}=\mathcal{N}\cap\{r<f(x')\}$and $\mathcal{N}^+=\{|D\Psi^{-1}D\varphi|<c\}=\mathcal{N}\cap\{r>f(x')\}$ for the (r,x') coordinates defined in (2.22), and find a corresponding subsonic solution (ρ_+,φ_+,p_+) in $\mathcal{N}^+$ so that*

(i) *(ρ_+,φ_+,p_+) satisfies (2.42)-(2.44) in $\mathcal{N}^+$;*

(ii) *On $\Gamma_w^+:=\Gamma_w\cap\partial\mathcal{N}^+$, φ_+ satisfies*

$$A(x,D\Psi,D\varphi)\cdot\nu_w=0 \tag{2.53}$$

for a unit normal ν_w on Γ_w;

(iii) *(φ_+,p_+) satisfies (2.47)-(2.49) on S;*

(iv) *$p=p_{ex}$ holds on Γ_{ex}.*

Our claim is that if a background supersonic solution $(\rho_0^-,\varphi_0^-,p_0^-)$ and a constant exit pressure $p_c\in(p_{min},p_{max})$ are perturbed sufficiently small, and also if Ψ is a small perturbation of the identity map, then there exists a corresponding transonic shock solution for a non-isentropic potential flow.

To study Problem 2, we will use weighted Hölder norms. For a bounded connected open set $\Omega\subset\mathbb{R}^n$, let Γ be a closed portion of $\partial\Omega$. For $x,y\in\Omega$, set

$$\delta_x:=dist(x,\Gamma)\quad \delta_{x,y}:=\min(\delta_x,\delta_y).$$

For $k\in\mathbb{R}$, $\alpha\in(0,1)$ and $m\in\mathbb{Z}_+$, we define

$$\begin{aligned}
&\|u\|^{(k,\Gamma)}_{m,0,\Omega}:=\sum_{0\le|\beta|\le m}\sup_{x\in\Omega}\delta_x^{\max(|\beta|+k,0)}|D^\beta u(x)|\\
&[u]^{(k,\Gamma)}_{m,\alpha,\Omega}:=\sum_{|\beta|=m}\sup_{x,y\in\Omega,x\neq y}\delta_{x,y}^{\max(m+\alpha+k,0)}\frac{|D^\beta u(x)-D^\beta u(y)|}{|x-y|^\alpha}\\
&\|u\|^{(k,\Gamma)}_{m,\alpha,\Omega}:=\|u\|^{(k,\Gamma)}_{m,0,\Omega}+[u]^{(k,\Gamma)}_{m,\alpha,\Omega}
\end{aligned}$$

where we write $D^\beta=\partial_{x_1}^{\beta_1}\cdots\partial_{x_n}^{\beta_n}$ for a multi-index $\beta=(\beta_1,\cdots,\beta_n)$ with $\beta_j\in\mathbb{Z}_+$ and $|\beta|=\sum_{j=1}^n\beta_j$. We use the notation $C^{m,\alpha}_{(k,\Gamma)}(\Omega)$ for the set of functions whose $\|\cdot\|^{(k,\Gamma)}_{m,\alpha,\Omega}$ norm is finite.

Remark 2.6. *Let Ω be a connected subset of $\mathcal{N}$, and Γ be a portion of $\partial\Omega$. For T in* (2.22)*, let us set $\Omega_*:=\{(r,x'):x=_T(r,x'),x\in\Omega\}$ and $\Gamma_*:=\{(r,x'):x=_T(r,x'),x\in\Gamma\}$. Then there is a constant $C_*(\mathcal{N},\Omega,n,m,\alpha,k)$ so that, for any $u\in C^{m,\alpha}_{k,\Gamma}(\Omega)$, there holds*

$$\frac{1}{C_*}\|u\|^{(k,\Gamma_*)}_{m,\alpha,\Omega_*}\le\|u\|^{(k,\Gamma)}_{m,\alpha,\Omega}\le C_*\|u\|^{(k,\Gamma_*)}_{m,\alpha,\Omega_*}. \tag{2.54}$$

For a fixed constant data $(\rho_{in},v_{in},p_{in})\in\mathbb{R}^3_+$ satisfying $v_{in}^2>\frac{\gamma p_{in}}{\rho_{in}}$ and $p_c\in(p_{min},p_{max})$, let (ρ_0,φ_0,p_0) be the background solution with $(\rho_0,\partial_r\varphi_0,p_0)(r_0)=(\rho_{in},v_{in},p_{in})$ and $p_0(r_1)=p_c$, and let $S_0=\{r=r_s\}\cap\mathcal{N}$ be the shock of (ρ_0,φ_0,p_0). Now, we state the main theorems.

Theorem 1 (*Existence*)**.** *For any given $\alpha\in(0,1)$, there exist constants $\sigma_1,\delta,C>0$ depending on $(\rho_{in},v_{in},p_{in}),p_c,r_0,r_1,\gamma,n,\Lambda$ and α so that whenever $0<\sigma\le\sigma_1$, if*

(i) *a diffeomorphism $\Psi:\overline{\mathcal{N}}\to\mathbb{R}^n$ satisfies*
$$\varsigma_1:=|\Psi-Id|_{2,\alpha,\mathcal{N}}\le\sigma,$$
(ii) *(ρ_-,φ_-,p_-) is a solution to* (2.42)-(2.44) *in $\mathcal{N}^-_{r_s+2\delta}$, and satisfies the boundary condition* (2.53) *on $\Gamma_w\cap\mathcal{N}^-_{r_s+2\delta}$ and the estimate*
$$\varsigma_2:=|\rho_--\rho_0^-|_{2,\alpha,\mathcal{N}^-_{r_s+2\delta}}+|p_--p_0^-|_{2,\alpha,\mathcal{N}^-_{r_s+2\delta}}+|\varphi_--\varphi_0^-|_{3,\alpha,\mathcal{N}^-_{r_s+2\delta}}\le\sigma,$$
(iii) *a function p_{ex} defined on Γ_{ex} satisfies*
$$\varsigma_3:=\|p_{ex}-p_c\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda}\le\sigma,$$

then there exists a transonic shock solution (ρ,φ,p) for a non-isentropic potential flow with a shock S satisfying that

(a) *there is a function $f:\Lambda\to\mathbb{R}^+$ so that S, $\mathcal{N}^-=\{x\in\mathcal{N}:|D\Psi^{-1}D\varphi|>c\}$ and $\mathcal{N}^+=\{x\in\mathcal{N}:|D\Psi^{-1}D\varphi|<c\}$ are given by*
$$\begin{aligned}&S=\{(r,x')\in\mathcal{N}:r=f(x'),x'\in\Lambda\},\\&\mathcal{N}^-=\mathcal{N}\cap\{(r,x'):r<f(x')\},\quad\mathcal{N}^+=\mathcal{N}\cap\{(r,x'):r>f(x')\},\end{aligned}$$
and f satisfies
$$|f-r_s|_{1,\alpha,\Lambda}\le C(\varsigma_1+\varsigma_2+\varsigma_3)\le C\sigma;$$
(b) *$(\rho,\varphi,p)=(\rho_-,\varphi_-,p_-)$ holds in $\mathcal{N}^-$;*

(c) $(\rho,\varphi,p)|_{\mathcal{N}^+}$ *satisfies* (2.47)-(2.49), (2.53) *and the estimate*

$$\|\rho-\rho_0^+\|^{(-\alpha,\Gamma_w)}_{1,\alpha,\mathcal{N}^+}+\|\varphi-\varphi_0^+\|^{(-1-\alpha,\Gamma_w)}_{2,\alpha,\mathcal{N}^+}+\|p-p_0^+\|^{(-\alpha,\Gamma_w)}_{1,\alpha,\mathcal{N}^+}\le C(\varsigma_1+\varsigma_2+\varsigma_3);$$

(d) $p=p_{ex}$ *holds on* Γ_{ex}.

Remark 2.7. *In this paper, we phrase any constant depending on* $\rho_{in}, v_{in}, p_{in}, p_c, r_0, r_1, \gamma, n, \Lambda$ *and* α *as a constant depending on the data unless otherwise specified.*

Theorem 2 (Uniqueness)**.** *Under the same assumptions with Theorem 1, for any* $\alpha\in(\frac{1}{2},1)$, *there exists a constant* $\sigma_2>0$ *depending on the data in sense of Remark 2.7 so that whenever* $0<\sigma\le\sigma_2$, *the transonic shock solution in Theorem 1 is unique in the class of weak solutions satisfying Theorem 1(a)-(d).*

Remark 2.8. *By Theorem 1(c), Theorem 2 implies that the transonic shock solutions are stable under a small and smooth perturbation of the incoming supersonic flow, exit pressure and the nozzle.*

2.6. **Framework of the proof of Theorem 1 and Theorem 2.** By the decomposition (2.42)-(2.44), if we solve (2.42) for φ, then we can plug it into the transport equation to solve for p, and solve the Bernoulli's law for ρ. To solve (2.43) for p, however, the initial condition on S can be computed by (2.48) and (2.49). But if we also fix $p=p_{ex}$ on Γ_{ex}, then the problem gets overdetermined. So, we set the framework to prove Theorem 1 and 2 as follows.

(Step 1) To prove Theorem 1, we first solve

Problem 3 (Free boundary problem for φ)**.** *Given the incoming supersonic flow* (ρ_-,φ_-,p_-) *and a function* v_{ex} *defined on* Γ_{ex}, *find* $f\in C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)$ *and* φ *in* $\mathcal{N}^+$ *so that (i)* φ *satisfies* $|D\Psi^{-1}D\varphi|<c$ *and* (2.42) *in* $\mathcal{N}^+=\mathcal{N}\cap\{(r,x'):r>f(x')\}$, *(ii) also satisfies* (2.47), (2.48), (2.53) *and* $M(x,D\varphi)=v_{ex}$ *on* Γ_{ex} *with the definition of* $M(x,D\varphi)$ *being given later.*

Let us emphasize that Lemma 2.5 is the key ingredient to prove Lemma 3.3, and this lemma provides well-posedness of Problem 3.

(Step 2) In Section 4, we solve (2.43) for p after plugging φ in to (2.43) by the method of characteristics in $\mathcal{N}^+$ with the initial condition (2.49) on S. However, as we will show later, $D\varphi$ is only in C^α up to the boundary of $\mathcal{N}^+$ so the standard ODE theory is not sufficient to claim solvability of (2.43). For that reason, we need a subtle analysis of $D\varphi$ to overcome this difficulty.

(Step 3) For a fixed (Ψ,φ_-,p_-), define $\mathcal{P}$ by $\mathcal{P}(\Psi,\varphi_-,p_-,v_{ex})=p_{ex}$ where p_{ex} is the exit value of the solution p to (2.43) where φ in Problem 3 is uniquely determined by $(\Psi,\varphi_-,p_-,v_{ex})$. In Section 5, we will show that $\mathcal{P}(\Psi,\varphi_-,p_-,\cdot):v_{ex}\mapsto p_{ex}$ is locally invertible in a weak sense near the background solution. The weak invertibility is proven by modifying the proof of the *right inverse mapping theorem* for finite dimensional Banach spaces in [SZ], and this proves Theorem 1.

(Step 4) To prove Theorem 2, we need to estimate the difference of two transonic shock solutions in a weaker space than the space of the transonic shock solutions in step 3. This requires a careful analysis of solutions to elliptic boundary problems with a blow-up of oblique boundary conditions at the corners of the nozzle $\mathcal{N}$. See Lemma 6.3.

3. Problem 3: Free Boundary Problem for φ

To solve Problem 3, we fix the boundary condition on the exit Γ_{ex} as

$$A(x, I_n, D\varphi)D\varphi \cdot \nu_{ex} = v_{ex} \tag{3.1}$$

where v_{ex} is a small perturbation of the constant v_c given by

$$v_c := A(x, I_n, D\varphi_0^+)D\varphi_0^+ \cdot \nu_{ex}|_{\Gamma_{ex}} = (B_0 - \frac{1}{2}|\partial_r\varphi_0^+(r_1)|^2)^{\frac{1}{\gamma-1}}\partial_r\varphi_0^+(r_1).$$

Here, ν_{ex} is the outward unit normal on Γ_{ex} so ν_{ex} is $\hat{r} = \frac{x}{|x|}$. Let (ρ_0, φ_0, p_0) and r_s be as in Theorem 1.

Proposition 3.1. *Under the assumptions of Theorem 1(i) and (ii), for any given $\alpha \in (0,1)$, there exist constants $\sigma_3, \delta, C > 0$ depending on the data in sense of Remark 2.7 so that whenever $0 < \sigma \le \sigma_3$, if a function v_{ex} defined on Γ_{ex} satisfies*

$$\varsigma_4 := \|v_{ex} - v_c\|_{1,\alpha,\Lambda}^{(-\alpha,\partial\Lambda)} \le \sigma,$$

then there exists a unique solution φ to (2.42), (2.47), (2.48), (2.53), (3.1) *with the shock S, and moreover,*

(a) *there exists a function $f : \Lambda \to \mathbb{R}^+$ so that S, $\mathcal{N}^- = \mathcal{N} \cap \{|D\Psi^{-1}D\varphi| > c\}$ and $\mathcal{N}^+ = \mathcal{N} \cap \{|D\Psi^{-1}D\varphi| < c\}$ are given by*

$$S = \{(r, x') \in \mathcal{N} : r = f(x'), x' \in \Lambda\},$$
$$\mathcal{N}^- = \mathcal{N} \cap \{(r, x') : r < f(x')\}, \quad \mathcal{N}^+ = \mathcal{N} \cap \{(r, x') : r > f(x')\},$$

and f satisfies

$$|f - r_s|_{1,\alpha,\Lambda} \le C(\varsigma_1 + \varsigma_2 + \varsigma_4) \le C\sigma,$$

(b) *$\varphi = \varphi_-$ holds in $\mathcal{N}^-$,*

(c) *φ satisfies the estimate*

$$\|\varphi - \varphi_0^+\|_{2,\alpha,\mathcal{N}^+}^{(-1-\alpha,\Gamma_w)} \le C(\varsigma_1 + \varsigma_2 + \varsigma_4) \le C\sigma. \tag{3.2}$$

We prove Proposition 3.1 by the implicit mapping theorem. For that, we first derive the equation and the boundary conditions for $\psi := \varphi - \varphi_0^+$.

3.1. Nonlinear boundary problem for $\psi = \varphi - \varphi_0^+$.

If φ is a solution in Proposition 3.1 then $\psi := \varphi - \varphi_0^+$ satisfies

$$\partial_k(a_{jk}(x, D\psi)\partial_j\psi) = \partial_k F_k(x, D\Psi, D\psi) \quad \text{in } \mathcal{N}^+, \tag{3.3}$$
$$(a_{jk}(x, D\psi)\partial_j\psi) \cdot \nu_w = [A(x, I_n, D\varphi) - A(x, D\Psi, D\varphi)] \quad \text{on } \Gamma_w^+, \tag{3.4}$$
$$(a_{jk}(x, D\psi)\partial_j\psi) \cdot \nu_{ex} = v_{ex} - v_c \quad \text{on } \Gamma_{ex} \tag{3.5}$$

with $\Gamma_w^+ := \Gamma_w \cap \partial\mathcal{N}^+$ and

$$a_{jk}(x, \eta) = \int_0^1 \partial_{\eta_k} A_j(x, I_n, D\varphi_0^+ + a\eta)da \tag{3.6}$$
$$F_k(x, m, \eta) = -A_k(x, m, D\varphi_0^+ + \eta) + A_k(x, I_n, D\varphi_0^+ + \eta) \tag{3.7}$$

where $A = (A_1, \cdots, A_n)$ is defined in (2.45). The boundary condition for ψ on S needs to be computed more carefully.

Remark 3.2. *In this paper, we fix the Bernoulli's invariant B as a constant B_0. So, for given (Ψ, φ_-, p_-), by (2.44), ρ_- is given by*

$$\rho_- = \frac{\gamma p_-}{(\gamma-1)(B_0 - \frac{1}{2}|d\Psi^{-1}D\varphi_-|^2)}.$$

So we regard (Ψ, φ_-, p_-) as factors of perturbations of the nozzle, and the incoming supersonic flow.

Lemma 3.3. *Let f and φ be as in Proposition 3.1 and set $\psi_- := \varphi_- - \varphi_0^-$. Then there are two functions $\mu = \mu_f$, $g_1 = g_1(x, D\Psi, p_-, \psi_-, D\psi_-, D\psi)$ in sense of Remark 3.2, and a vector valued function $b_1 = b_1(x, D\psi_-, D\psi)$ so that ψ satisfies*

$$b_1 \cdot D\psi - \mu_f \psi = g_1 \text{ on } S. \tag{3.8}$$

Moreover, b_1, μ_f and g_1 satisfy, for $\mu_0 > 0$ defined in Lemma 2.5,

$$\|\mu_f - \mu_0\|_{1,\alpha,\Lambda} \le C(\varsigma_1 + \varsigma_2 + \varsigma_4), \tag{3.9}$$

$$\|b_1 - \nu_s\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda} \le C(\varsigma_1 + \varsigma_2 + \varsigma_4), \tag{3.10}$$

$$\|g_1\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda} \le C(\varsigma_1 + \varsigma_2). \tag{3.11}$$

So if σ_3 (in Proposition 3.1) is sufficiently small depending on the data, then there is a constant $\lambda > 0$ satisfying

$$\mu_f \ge \frac{\mu_0}{2} > 0, \quad \text{and} \quad b_1 \cdot \nu_s \ge \lambda \quad \text{on} \quad S \tag{3.12}$$

where the constant λ also only depends on the data as well.

Proof. **Step 1.** In general, at $(f(x'), x')$ for $x' \in \Lambda$, $\psi = \varphi - \varphi_0^+$ satisfies

$$\partial_r \psi = \partial_r \varphi - \frac{K_0}{\partial_r \varphi_0^-} + (\frac{K_0}{\partial_r \varphi_0^-} - \partial_r \varphi_0^+) \tag{3.13}$$

where K_0 is given by (2.30). By (2.32) and (2.47), (3.13) is equivalent to

$$D\psi \cdot \hat{r} - \mu_f \psi = h \tag{3.14}$$

with

$$\mu_f(x') := \frac{\int_0^1 \frac{d}{dr}\big(\frac{K_0}{\partial_r \varphi_0^-} - \partial_r \varphi_0^+\big)(r_s + t(f(x') - r_s))dt}{\int_0^1 \partial_r(\varphi_0^- - \varphi_0^+)(r_s + t(f(x') - r_s))dt}, \tag{3.15}$$

$$h = D\varphi \cdot \hat{r} - \frac{K_0}{D\varphi_0^- \cdot \hat{r}} - \mu_f(\varphi_- - \varphi_0^-) =: h_* - \mu_f \psi_- \tag{3.16}$$

where we set $\psi_- =: \varphi_- - \varphi_0^-$.

Suppose that $\Psi = Id, (\varphi_-, p_-) = (\varphi_0^-, p_0^-)$ and φ in Proposition 3.1 is a radial function with a shock on $S = \{r = f_0\}$ for a constant $f_0 \in (r_0, r_1)$. Then, (2.32) implies $h = 0$ in (3.14). Then the boundary condition for ψ on S is

$$\partial_r \psi - \mu_{f_0} \psi = 0 \text{ on } S = \{r = f_0\}. \tag{3.17}$$

Step 2. (3.17) shows how to formulate a boundary condition for φ in general.

Since $\frac{K_0}{\partial_r \varphi_0^-} - \partial_r \varphi_0^+$ is smooth, by Proposition 3.1, we have

$$\|\mu_f - \mu_0\|_{1,\alpha,\Lambda} \le C\|f - r_s\|_{1,\alpha,\Lambda} \le C\|\psi - \psi_-\|_{1,\alpha,S} \le C(\varsigma_1 + \varsigma_2 + \varsigma_4).$$

(3.9) is verified. By Lemma 2.5, μ_0 is a positive constant.

We note that h^* in (3.16) depends on $D\Psi, p_-, D\psi_-$ in sense of Remark 3.2, and also depends on $D\psi$. So we should carefully decompose h^* in the form of $h^* = \beta \cdot D\psi + g$ so that a weighted Hölder norm of g does not depend on $\|\psi\|^{(-1-\alpha,\Gamma_w)}_{2,\alpha,\mathcal{N}^+}$ if σ_3 is sufficiently small. For h^* defined in (3.16), by (2.48), we set $h^* = a_1 - a_2$ with

$$a_1 = D\varphi \cdot \hat{r} - QD\varphi \cdot \nu_s, \quad a_2 = \frac{K_0}{D\varphi_0^- \cdot \hat{r}} - \frac{K_s}{QD\varphi^- \cdot \nu_s} \text{ on } S. \tag{3.18}$$

We note that, by (2.47), $\nu_s - \hat{r}$ is written as

$$\nu_s - \hat{r} = \nu(D\psi_-, D\psi) - \nu(0,0) = j_1(D\psi_-, D\psi)D\psi_- + j_2(D\psi_-, D\psi)D\psi \tag{3.19}$$

where we set

$$\nu(\xi,\eta) := \frac{(D\varphi_0^- + \xi) - (D\varphi_0^+ + \eta)}{|(D\varphi_0^- + \xi) - (D\varphi_0^+ + \eta)|}, \tag{3.20}$$

$$j_1(\xi,\eta) := \int_0^1 D_\xi \nu(t\xi,\eta)dt\ , \quad j_2(\xi,\eta) := \int_0^1 D_\eta \nu(0,t\eta)dt \tag{3.21}$$

for $\xi,\eta \in \mathbb{R}^n$. Then, we can rewrite a_1 as

$$\begin{aligned} a_1 &= D\varphi \cdot (\hat{r} - \nu_s) + (I - Q)D\varphi \cdot \nu_s \\ &= -[j_2^T(D\psi_-, D\psi)]D(\varphi_0^+ + \psi) \cdot D\psi + h_1 := \beta_1 \cdot D\psi + h_1 \end{aligned} \tag{3.22}$$

with

$$h_1(D\Psi, D\psi_-, D\psi) = -D\varphi \cdot [j_1(D\psi_-, D\psi)D\psi_-] - (Q - I)D\varphi \cdot \nu_s.$$

Here, all the quantities are evaluated on $(f(x'), x') \in S$.

To rewrite a_2 for our purpose, let us consider $K_0 - K_s$ first.

$$K_0 - K_s = \frac{\gamma - 1}{\gamma + 1}(\underbrace{(D\varphi_0^- \cdot \hat{r})^2 - (QD\varphi_- \cdot \nu_s)^2}_{=:q_*}) + \frac{2\gamma}{\gamma + 1}(\frac{p_-}{\rho_-} - \frac{p_0^-}{\rho_0^-}) \tag{3.23}$$

$\|(\frac{p_-}{\rho_-} - \frac{p_0^-}{\rho_0^-})|_S\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda}$ is independent of φ but q_* should be treated carefully.

$$q_* = |D\varphi_0^-|^2(1 - (\hat{r} \cdot \nu_s)^2) + [(D\varphi_0^- \cdot \nu_s)^2 - (QD\varphi_- \cdot \nu_s)^2] =: I_1 + I_2$$

It is easy to see that a Hölder norm of I_2 is uniformly bounded independent of φ. Let us consider I_1 now. By (2.52), a simple calculation provides

$$1 - (\hat{r} \cdot \nu_s)^2 = \frac{|D(\psi_- - \psi)|^2 - |\partial_r(\psi_- - \psi)|^2}{|D(\varphi_- - \varphi)|^2} =: \frac{(D\psi - \partial_r\psi\hat{r}) \cdot D\psi}{|D(\varphi_- - \varphi)|^2} + I_3 \tag{3.24}$$

where a Hölder norm of I_3 is independent of ψ if σ_3 is sufficiently small. By (3.18), (3.23) and (3.24), a_2 is

$$\begin{aligned} a_2 &= (\frac{(\gamma - 1)\partial_r\varphi_0^-(D\psi - \partial_r\psi\hat{r})}{(\gamma + 1)|D(\varphi_- - \varphi)|^2} + \frac{[j_2^T(D\psi_-, D\psi)]K_s\hat{r}}{QD\varphi_- \cdot \nu_s}) \cdot D\psi + h_2 \\ &=: \beta_2 \cdot D\psi + h_2 \end{aligned} \tag{3.25}$$

with

$$\begin{aligned} h_2 = &\frac{\gamma-1}{\gamma+1}(\partial_r\varphi_0^- I_3 + I_2) + \frac{2\gamma}{(\gamma+1)\partial_r\varphi_0^-}(\frac{p_-}{\rho_-} - \frac{p_0^-}{\rho_0^-}) \\ &+ \frac{K_s}{QD\varphi_-\cdot\nu_s}(\frac{(Q-I)D\varphi_-\cdot\nu_s + D\psi_-\cdot\nu_s}{\partial_r\varphi_0^-} + [j_1^T(D\psi_-, D\psi)]\partial_r\psi_-). \end{aligned} \tag{3.26}$$

By (3.22) and (3.26), we can write $h^* = a_1 - a_2$ as

$$h^* = (\beta_1 - \beta_2)\cdot D\psi + (h_1 - h_2) \tag{3.27}$$

for $\beta_{1,2} = \beta_{1,2}(D\psi_-, D\psi), h_{1,2} = h_{1,2}(D\Psi, D\psi_-, D\psi)$ defined in (3.22) and (3.25). From this, we define b_1, g_1 in (3.8) by

$$b_1 = \hat{r} - \beta_1 + \beta_2, \quad g_1 = h_1 - h_2 - \mu_f\psi_-. \tag{3.28}$$

By the definition of g_1 in (3.28) and Proposition 3.1, we have

$$\|g_1(D\Psi, p_-, D\psi_-, \psi_-, D\psi)\|_{1,\alpha,\Lambda}^{(-\alpha,\partial\Lambda)} \le C(1+\sigma)(\varsigma_1 + \varsigma_2)$$

so (3.11) holds true if we choose σ_3 satisfying $0 < \sigma_3 \le 1$.

It remains to verify (3.10). From the definition of $\nu(\xi,\eta)$, one can check

$$D_\eta\nu(0,0) = \frac{I_n}{\partial_r(\varphi_0^- - \varphi_0^+)} - \frac{D^T(\varphi_0^- - \varphi_0^+)D(\varphi_0^- - \varphi_0^+)}{|D(\varphi_0^- - \varphi_0^+)|^3}$$

so we have $j_2(0,0)\hat{r} = 0$ where j_2 is defined in (3.21) then (3.22) and (3.25) imply $\beta_1(0,0) = \beta_1(0,0) = \vec{0}$. As vector valued functions of $\xi, \eta \in \mathbb{R}^n$, β_1 and β_2 are smooth for $\xi, \eta \in B_d^n(0)$ for a constant $d > 0$ sufficiently small. So if σ_3 is sufficiently small, then we obtain

$$\|\beta_{1,2}(D\psi_-, D\psi)\|_{1,\alpha,\Lambda}^{(-\alpha,\partial\Lambda)} \le C(\varsigma_1 + \varsigma_2 + \varsigma_4), \tag{3.29}$$

and this implies

$$\|b_1(D\psi_-, D\psi) - \nu_s\|_{1,\alpha,\Lambda}^{(-\alpha,\partial\Lambda)} \le C(\varsigma_1 + \varsigma_2 + \varsigma_4) + \|\nu_s - \hat{r}\|_{1,\alpha,\Lambda}^{(-\alpha,\partial\Lambda)}.$$

By (2.52) and Proposition 3.1, $\nu_s - \hat{r}$ satisfies

$$\|\nu_s - \hat{r}\|_{1,\alpha,\Lambda}^{(-\alpha,\partial\Lambda)} \le C(\varsigma_1 + \varsigma_2 + \varsigma_4),$$

therefore (3.10) is verified provided that we choose σ_3 in Proposition 3.1 sufficiently small depending on the data. □

3.2. **Proof of Proposition 3.1.** Fix $\alpha \in (0,1)$. For a constant $R > 0$, keeping Remark 3.2 in mind, we define neighborhoods of Id, (φ_0^-, p_0^-), v_c, r_s.

$$\begin{aligned} &\mathcal{B}_R^{(11)}(Id) := \{\Psi \in C^{2,\alpha}(\overline{\mathcal{N}}, \mathbb{R}^n) : |\Psi - Id|_{2,\alpha,\mathcal{N}} \le R\}, \\ &\mathcal{B}_R^{(12)}(\varphi_0^-, p_0^-) := \{(\varphi_-, p_-) \in C^{3,\alpha}(\overline{\mathcal{N}_{r_s+\delta}^-}) \times C^{2,\alpha}(\overline{\mathcal{N}_{r_s+\delta}^-}) : \\ &\qquad\qquad |\varphi^- - \varphi_0^-|_{3,\alpha,\mathcal{N}_{r_s+\delta}^-} + |p^- - p_0^-|_{2,\alpha,\mathcal{N}_{r_s+\delta}^-} \le R\}, \\ &\mathcal{B}_R^{(13)}(v_c) := \{v_{ex} \in C_{(-\alpha,\partial\Lambda)}^{1,\alpha}(\Lambda) : \|v_{ex} - v_c\|_{1,\alpha,\Lambda}^{(-\alpha,\partial\Lambda)} \le R\}, \\ &\mathcal{B}_R^{(1)}(Id, \varphi_0^-, p_0^-, v_c) := \mathcal{B}_R^{(11)}(Id) \times \mathcal{B}_R^{(12)}(\varphi_0^-, p_0^-) \times \mathcal{B}_R^{(13)}(v_c), \\ &\mathcal{B}_R^{(2)}(r_s) := \{f \in C_{(-1-\alpha,\partial\Lambda)}^{2,\alpha}(\Lambda) : \|f - r_s\|_{2,\alpha,\Lambda}^{(-1-\alpha,\partial\Lambda)} \le R\}. \end{aligned} \tag{3.30}$$

For a sufficiently small $R>0$, $\mathcal{B}^{(1)}_R(Id,\varphi_0^-,p_0^-,v_c)$ is a set of small perturbations of the nozzle, the incoming supersonic flow and the exit boundary condition for φ. For two constants $C^*,\sigma^*>0$, define $\mathfrak{J}:\mathcal{B}^{(1)}_{\sigma^*}(Id,\varphi_0^-,p_0^-,v_c)\times\mathcal{B}^{(2)}_{C^*\sigma^*}(r_s)\to C^{2,\alpha}_{(-1-\alpha,\partial\Lambda)}(\Lambda)$ by

$$\mathfrak{J}(\Psi,\varphi_-,p_-,v_{ex},f):=(\varphi_--\varphi_0^+-\psi)(f(x'),x') \tag{3.31}$$

for $x'\in\Lambda$ where ψ satisfies the following conditions: setting

$$\mathcal{N}_f^+:=\mathcal{N}\cap\{r>f(x')\},\quad S_f:=\{r=f(x'),x'\in\Lambda\},$$

(i) ψ satisfies
$$\|\psi\|^{(-1-\alpha,\Gamma_w)}_{2,\alpha,\mathcal{N}_f^+}\le M\sigma^* \tag{3.32}$$
for a constant $M>0$ to be determined later;

(ii) ψ solves the nonlinear boundary problem:
$$\begin{aligned}
&N_+(\psi):=\sum_{j,k}\partial_k(a_{jk}(x,D\psi)\partial_j\psi)=\sum_k\partial_kF_k(x,D\Psi,D\psi)\quad\text{in}\quad\mathcal{N}_f^+\\
&M_f^1(\psi):=b_1(D\psi,\psi)\cdot D\psi-\mu_f\psi\\
&\qquad\quad=g_1(x,D\Psi,p_-,\psi_-,D\psi_-,D\psi)\quad\text{on}\quad S_f\\
&M^2(\psi):=[a_{jk}(x,D\psi)]D\psi\cdot\nu_w=g_2(x,D\Psi,D\psi)\quad\text{on}\quad\Gamma_{w,f}:=\partial\mathcal{N}_f^+\cap\Gamma_w\\
&M^3(\psi):=[a_{jk}(x,D\psi)]D\psi\cdot\nu_{ex}=g_3(v_{ex})\quad\text{on}\quad\Gamma_{ex}
\end{aligned}\tag{3.33}$$
with
$$\begin{aligned}
&g_2(x,D\Psi,D\psi)=-[A(x,D\Psi,D\varphi_0^++D\psi)-A(x,I_n,D\varphi_0^++D\psi)],\\
&g_3(v_{ex})=v_{ex}-v_c
\end{aligned}\tag{3.34}$$

where A in (3.34) is defined as in (3.1). For $\sigma^*>0$ sufficiently small, if M in (3.32) is appropriately chosen depending on the data so that (3.33) is a uniformly elliptic boundary problem with oblique boundary conditions, then $\mathfrak{J}$ is well defined. More precisely, we have the following lemma.

Lemma 3.4 (Well-posedness (3.33)). *Fix $\alpha\in(0,1)$ and $C^*>0$. Then there is a constant $M,\sigma^\sharp$ depending on the data in sense of Remark 2.7 and also depending on C^* so that the followings hold:*

(i) *for any given $(\Psi,\varphi_-,p_-,v_{ex})\in\mathcal{B}^{(1)}_{\sigma^\sharp}(Id,\varphi_0^-,p_0^-,v_c)$, $f\in\mathcal{B}^{(2)}_{C^*\sigma^\sharp}(r_s)$, (3.33) is a uniformly elliptic boundary problem, and it has a unique solution ψ satisfying (3.32),*

(ii) *if $\sigma^\sharp$ is sufficiently small depending on the data and C^*, then there is a constant C depending only on the data so that the solution of (3.33) satisfies*
$$\|\psi\|^{(-1-\alpha,\Gamma_w)}_{2,\alpha,\mathcal{N}_f^+}\le C(\varsigma_1+\varsigma_2+\varsigma_4)$$
where $\varsigma_1,\varsigma_2,\varsigma_4$ are defined in Theorem 2.4 and Proposition 3.1.

We will prove Lemma 3.4 by the contraction mapping principle. For that purpose, we first need to study a linear boundary problem related to (3.33). Let $\sigma^*>0$ be a small constant to be determined later. For a given constant $C^*>0$, fix $(\Psi,\varphi_-,p_-,v_{ex})\in\mathcal{B}^{(1)}_{\sigma^*}(Id,\varphi_0^-,p_0^-,v_c)$,

$f \in \mathcal{B}^{(2)}_{C^*\sigma^*}(r_s)$, and let us set $\psi_- = \varphi_- - \varphi_0^-$. Reduce σ^* sufficiently small depending on the data so that $D\Psi$ is invertible in $\mathcal{N}$, and set

$$\mathcal{K}_f(M) := \{\phi \in C^{1,\alpha}(\overline{\mathcal{N}_f^+}) : \|\phi\|^{(-1-\alpha,\Gamma_w)}_{2,\alpha,\mathcal{N}_f^+} \le M\sigma^*\}.$$

For a fixed $\phi \in \mathcal{K}_f(M)$, consider the following linear problem in $\mathcal{N}_f^+$:

$$\begin{aligned}
&\sum_{j,k} \partial_k(a_{jk}(x, D\phi)\partial_j u) = \sum_k \partial_k F_k(x, D\Psi, D\phi) \quad \text{in} \quad \mathcal{N}_f^+,\\
&b_1(D\phi, \phi) \cdot Du - \mu_f u = g_1(x, D\Psi, p_-, \psi_-, D\psi_-, D\phi) \quad \text{on} \quad S_f,\\
&(a_{jk}(x, D\phi)\partial_j u) \cdot \nu_w = g_2(x, D\Psi, D\phi) \quad \text{on} \quad \partial\mathcal{N}_f^+ \cap \Gamma_w,\\
&(a_{jk}(x, D\phi)\partial_j u) \cdot \nu_{ex} = g_3(v_{ex}) \quad \text{on} \quad \Gamma_{ex}.
\end{aligned} \tag{3.35}$$

with $\psi_- := \varphi_- - \varphi_0^-$. The definitions of $a_{jk}, b_1, \mu_f, g_1, g_2, g_3$ are given in (3.6), (3.15), (3.28), and (3.34).

Lemma 3.5 (Well posedness of (3.35))**.** *For any given $\alpha \in (0,1)$, and $M, C^* > 0$ is $\sigma^* > 0$ is chosen sufficiently small depending only on the data, C^* and M, then there exists a constant C depending on the data and α so that, for any given $f \in \mathcal{B}^{(2)}_{C^*\sigma^*}(r_s)$ and $\phi \in \mathcal{K}_f(M)$, (3.35) has a unique solution $u^{(\phi)} \in C^{2,\alpha}_{(-1-\alpha,\Gamma_w)}(\mathcal{N}_f^+)$. Moreover $u^{(\phi)}$ satisfies*

$$\|u^{(\phi)}\|^{(-1-\alpha,\Gamma_w)}_{2,\alpha,\mathcal{N}_f^+} \le C(\varsigma_1 + \varsigma_2 + \varsigma_4). \tag{3.36}$$

To prove Lemma 3.5, we basically repeat the proof of [CF3, Theorem 3.1]. A key ingredient of the proof in [CF3] is the method of continuity and the divergence structure of a linear boundary problem. In our case, however, (3.35) is not in a divergence form because of the derivative boundary condition on S_f. For that reason, we need Lemma 3.8 so that (3.35) can be treated as a divergence problem. Before we proceed further, we first note two obvious lemmas for the later use.

Lemma 3.6. *For any $C^*, M > 0$, and for $f \in \mathcal{B}^{(2)}_{C^*\sigma^*}(r_s), \phi \in \mathcal{K}_f(M)$, there hold*

$$\|a_{jk}(x, D\phi) - a_{jk}(x, 0)\|^{(-\alpha,\Gamma_w)}_{1,\alpha,\mathcal{N}_f^+} \le CM\sigma^*, \tag{3.37}$$

$$\|b_1(D\phi, \phi) - b_1(0, 0)\|^{(-\alpha,\Gamma_w)}_{1,\alpha,\mathcal{N}_f^+} \le CM\sigma^*, \tag{3.38}$$

$$\|\mu_f - \mu_0\|_{1,\alpha,\Lambda} \le C\|f - r_s\|_{1,\alpha,\Lambda}, \tag{3.39}$$

$$\begin{aligned}
&\|F(x, D\Psi)\|^{(1-\alpha,\Gamma_w)}_{\alpha,\mathcal{N}_f^+} + \|g_1(x, D\Psi, p_-, \psi_-, D\psi_-, D\phi)\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda}\\
&\qquad + \|g_2(x, D\Psi, D\phi)\|_{\alpha,\Gamma_{w,f}} + \|g_3(v_{ex})\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda} \le C\sigma^*.
\end{aligned} \tag{3.40}$$

where the constant C in (3.37)-(3.40) *only depends on the data in sense of Remark 2.7. Thus, for any $C^*, M > 0$, if σ^* is chosen sufficiently small depending on the data, C^* and M, then the equation of* (3.35) *is uniformly elliptic in $\mathcal{N}_f^+$.*

Proof. (3.37)-(3.40) are obvious by the definitions.

For any given $\zeta \in \mathbb{R}^n$, $a_{ij}(x, 0)$ satisfies

$$a_{ij}(x, 0)\zeta_i\zeta_j \ge \frac{(\gamma + 1)}{2(\gamma - 1)}(B_0 - \frac{1}{2}|D\varphi_0^+|^2)^{\frac{2-\gamma}{\gamma-1}}(K_0 - |D\varphi_0^+|^2)|\zeta|^2. \tag{3.41}$$

So, by (2.34), (2.35) and Remark 2.2, for any $C^*, M > 0$, one can choose σ^* depending on the data, C^* and M so that (3.35) is uniformly elliptic in $\mathcal{N}_f^+$. $\square$

Lemma 3.7. *For any $C^*, M > 0$, let ν_f be the inward unit normal on S_f, and b_1 be defined in* (3.28)*. Then, for any givenf $\in \mathcal{B}^{(2)}_{C^*\sigma^*}(r_s)$, $\phi \in \mathcal{K}_f(M)$, we have*

$$b_1(D\phi, \phi) \cdot \nu_f \geq \frac{1}{2} \quad on \quad S_f, \tag{3.42}$$

$$\mu_f \geq \frac{1}{2}\mu_0 > 0 \quad on \quad S_f \quad for\ any\ f \in \mathcal{B}^{(2)}_{C^*\sigma^*}(r_s) \tag{3.43}$$

where μ_0 is defined in Lemma 2.5 if σ^* is chosen sufficiently small depending on the data, C^* and M.

Proof. For ν defined in (3.20), we have

$$\begin{aligned} & b_1(D\phi, \phi) \cdot \nu_f \\ &= (b_1(D\phi, \phi) - b_1(0,0)) \cdot \nu_f + b_1(0,0) \cdot (\nu_f - \nu(0,0)) + b_1(0,0) \cdot \nu(0,0) \\ &\geq -C(M + C^*)\sigma^* + 1 \end{aligned}$$

for a constant C depending only on the data. By (3.38), for any $C^*, M > 0$, if we choose σ^* to satisfy $-C(M + C^*)\sigma^* + 1 \geq \frac{1}{2}$, then (3.42) is obtained. Reducing σ^* if necessary, (3.40) provides (3.43). $\square$

By Lemma 3.6 and 3.7, for any $C^*, M > 0$, choosing σ^* sufficiently small depending on the data, C^* and M, (3.3) is a uniformly elliptic linear boundary problem with oblique boundary conditions on the boundary.

Proof of Lemma 3.5.
Step 1. Suppose that u solves (3.35) and then we first verify (3.36) by following the method of the proof for [CF3, Theorem 3.1]. First, we apply the change of variables $x \mapsto y = \Phi(x)$ with Φ defined by

$$|y| = r_s + \frac{r_1 - r_s}{r_1 - f(x')}(|x| - f(x')), \qquad \frac{y}{|y|} = \frac{x}{|x|}, \tag{3.44}$$

This change of variables transforms $\mathcal{N}_f^+$ to $\mathcal{N}_{r_s}^+$, and $S_f = \{r = f(x')\} \cap \mathcal{N}_f^+$ to $S_0 = \{r = r_s\} \cap \mathcal{N}_{r_s}^+$ in the spherical coordinates in (2.22). Then $w(y) = u \circ \Phi^{-1}(y)$ solves a linear boundary problem of the same structure as (3.35). For simplicity, we write the the equation and the boundary conditions for w as

$$\begin{aligned} &\partial_k(\tilde{a}_{jk}(y, f, D\phi)\partial_j w) = \partial_k \tilde{F}_k(y, f, D\Psi, D\phi) \ \text{ in } \ \mathcal{N}_{r_s}^+, \\ &\tilde{b}_1(y, f, \phi, D\phi) \cdot Dw - \mu_f w = \tilde{g}_1(y, f, D\Psi, p_-, \psi_-, D\psi_-, D\phi) \ \text{ on } \ S_0 \\ &(\tilde{a}_{jk}(y, f, D\phi)\partial_j w) \cdot \nu_w = \tilde{g}_2(D\Psi, f, D\phi) \ \text{ on } \ \Gamma_{w,r_s}, \\ &(\tilde{a}_{jk}(y, f, D\phi)\partial_j w) \cdot \nu_{ex} = \tilde{g}_3(v_{ex}) \ \text{ on } \ \Gamma_{ex}. \end{aligned} \tag{3.45}$$

with $\Gamma_{w,r_s} = \Gamma_w \cap \partial\mathcal{N}_{r_s}^+$ and $\psi_- = \varphi_- - \varphi_0^-$, where $\tilde{a}_{jk}, \tilde{b}_1, \tilde{F}_k, \tilde{g}_m$ are obtained from a_{jk}, b_1, F_k, g_m in (3.35) through the change of variables $x \mapsto y = \Phi(x)$ for $j, k = 1, \cdots, n$, and $m = 1, 2, 3$. Next, we rewrite (3.45) as a conormal boundary problem with coefficients $a_{jk}(y, 0)$ and $b_1(y, 0) = \hat{r}$ for the equation and the boundary conditions of w.

We remind that, by (3.6), the matrix $[a_{jk}(x,0)]$ is

$$[a_{jk}(y,0)] = (B_0 - \frac{1}{2}|D\varphi_0^+|^2)^{\frac{1}{\gamma-1}} I_n - \frac{(B_0 - \frac{1}{2}|D\varphi_0^+|^2)^{\frac{2-\gamma}{\gamma-1}}}{\gamma-1}(D\varphi_0^+)^T D\varphi_0^+. \tag{3.46}$$

Lemma 3.8 (Boundary condition for w on S_0)**.** *Suppose that w solves* (3.45)*, and let M, C^*, σ^* be as in Lemma 3.5. For any given $\alpha \in (0,1)$, $f \in B^{(2)}_{C^*\sigma^*}(r_s)$, $\phi \in \mathcal{K}_f(M)$, there is a constant C depending only on the data in sense of Remark 2.7 so that w satisfies*

$$(a_{jk}(y,0)\partial_j w)\cdot \nu_s = h_1 \quad on \;\; S_0 \tag{3.47}$$

for a function $h_1 \in C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)$, and h_1 satisfies

$$\|h_1\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda} \le C((M+C^*)\sigma^*\|w\|^{(-1-\alpha,\Gamma_w)}_{2,\alpha,\mathcal{N}^+_{r_s}} + \|w\|^{(-\alpha,\Gamma_w)}_{1,\alpha,\mathcal{N}^+_{r_s}} + \|\tilde{g}_1\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda}). \tag{3.48}$$

Proof. The unit normail on S_0 toward $\mathcal{N}^+_{r_s}$ is $\hat{r}$. Then, by (3.46) and the boundary condition of w on S_0 in (3.45), there holds

$$\begin{aligned}(a_{jk}(y,0)\partial_j u)\cdot\nu_s &= (B_0 - \frac{1}{2}|D\varphi_0^+|^2)^{\frac{1}{\gamma-1}}(1 - \frac{|D\varphi_0^+|^2}{(\gamma-1)(B_0-\frac{1}{2}|D\varphi_0^+|^2)})\partial_r w\\ &= \frac{(\gamma-1)B_0 - \frac{\gamma+1}{2}|D\varphi_0^+|^2}{(\gamma-1)(B_0-\frac{1}{2}|D\varphi_0^+|^2)^{1-\frac{1}{\gamma-1}}}[(\hat{r}-\tilde{b}_1)\cdot Dw + \mu_f w + \tilde{g}_1] =: h_1\end{aligned}$$

By (3.38) and the definition of $\tilde{b}_1$, (3.48) is easily obtained. □

Back to the proof of Lemma 3.5, using Lemma 3.8, w solves the conomal boundaru problem

$$\begin{aligned}&\partial_k(a_{jk}(y,0)\partial_j w) = \partial_k[\tilde{F}_k(y,f,D\Psi,D\phi) + \delta\tilde{a}_{jk}\partial_j w] =: \partial_k G_k \;\text{ in }\; \mathcal{N}^+_{r_s},\\ &(a_{jk}(y,0)\partial_j u)\cdot\nu_s = h_1 \;\text{ on }\; S_0,\\ &\partial_k(a_{jk}(y,0)\partial_j w)\cdot\nu_w = \tilde{g}_2 + (\delta\tilde{a}_{jk}\partial_j w)\cdot\nu_w =: h_2 \;\text{ on }\; \Gamma_{w,r_s},\\ &\partial_k(a_jk(y,0)\partial_j w)\cdot\nu_{ex} = \tilde{g}_3 + (\delta\tilde{a}_{jk}\partial_j w)\cdot\nu_{ex} =: h_3 \;\text{ on }\; \Gamma_{ex}\end{aligned} \tag{3.49}$$

with $\delta\tilde{a}_{jk} := a_{jk}(y,0) - \tilde{a}_{jk}(y,f,D\phi)$. We note that, by (3.37), $\delta\tilde{a}_{jk}$ satisfies

$$\|\delta\tilde{a}_{jk}\|^{(-\alpha,\Gamma_w)}_{1,\alpha,\mathcal{N}^+_{r_s}} \le C(\|\phi\|^{(-1-\alpha,\Gamma_w)}_{2,\alpha,\mathcal{N}^+_f} + \|f-r_s\|^{(-1-\alpha,\partial\Lambda)}_{2,\alpha,\Lambda}) \tag{3.50}$$

for a constant C depending on the data in sense of Remark 2.7.

Step 2. Using a weak formulation of the equation for w in (3.49) with (3.46), one can check

$$\begin{aligned}&\partial_k(a_{jk}(y,0)\partial_j w) = \partial_k G_k\\ &\Leftrightarrow L_0 := \partial_r(k_1(r)\omega_1(x')\partial_r w) + \partial_{x'_l}(k_2(r)b_{lm}(x')\partial_{x'_m} w) = \partial_r(\tilde{G}\cdot\hat{r}) + \sum_{l=1}^{n-1}\partial_{x'_l}(\tilde{G}\cdot\hat{x}'_l)\end{aligned} \tag{3.51}$$

with

$$\begin{aligned}&k_1(r) := \frac{(\gamma+1)r^{n-1}(K_0 - (\partial_r\varphi_0^+(r))^2)}{2(\gamma-1)(B_0 - \frac{1}{2}(\partial_r\varphi_0^+(r))^2)^{\frac{\gamma-2}{\gamma-1}}}, \;\; k_2(r) := r^{n-3}(B_0 - \frac{1}{2}(\partial_r\varphi_0^+)^2)^{\frac{1}{\gamma-1}}\\ &r^{n-1}\omega_1(x') = \det\frac{\partial x}{\partial(r,x')}\end{aligned} \tag{3.52}$$

for (r, x') is given as in (2.22), and every $b_{lm}(x')$ is smooth in Λ. Here we set $\tilde{G} = \omega_1[\frac{\partial(r,x')}{\partial x}]^T G$.

We note that ω_1 is bounded below by a positive constant in $\overline{\Lambda}$, and the linear operator L_0 is uniformly elliptic in $\mathcal{N}^+_{r_s}$,and bounded by (3.41) and (2.35).

Step 3. To obtain the desired estimate of u in (3.35), we use (3.49)-(3.52), Lemma 3.8 and the interpolation inequality of Hölder norms to follow the method of the proof for [CF3, Theorem 3.1], and prove the following lemma.

Lemma 3.9. *There exists a constant $\kappa > 0$ depending only on the data in sense of Remark 2.7 so that whenever $M\sigma^*, C^*\sigma^* \leq \kappa$, for any $f \in B^{(2)}_{\sigma}(r_s)$, $\phi \in \mathcal{K}_f(M)$ with $\alpha \in (0,1)$, there is a constant C depending on the data so that, if w solves* (3.45) *then there holds*

$$|w|_{1,\alpha,\mathcal{N}^+_{r_s}} \leq C(|w|_{0,\mathcal{N}^+_{r_s}} + |\tilde{F}|_{0,\alpha,\mathcal{N}^+_{r_s}} + |\tilde{g}_1|_{0,\alpha,S_0} + |\tilde{g}_2|_{0,\alpha,\Gamma_{w,r)s}} + |\tilde{g}_3|_{0,\alpha,\Gamma_{ex}}). \tag{3.53}$$

The only difference to prove Lemma (3.9) from [CF3, Theorem3.1] is that, to apply the method of reflection as in [CF3, Theorem3.1], we need to extend $a_{jk}(y,0)$ in r-direction by reflection on S_0 and Γ_{ex}. For that purpose, we use (3.52). Except for that, the proof of Lemma 3.9 is basically same as the proof for [CF3, Theorem3.1] so we skip details.

Once (3.53) is obtained, then the standard scaling provides $C^{2,\alpha}_{(-1-\alpha,\Gamma_w)}$ estimate of w. For a fixed point $y_0 \in \overline{\mathcal{N}^+_{r_s}} \setminus \Gamma_{w,r_s}$ let $2d := dist(y_0, \Gamma_{w,r_s})$, and we set a scaled function $w^{(y_0)}(s)$ near $y = y_0$ by

$$w^{(y_0)}(s) := \frac{w(y_0 + ds) - w(y_0)}{d^{1+\alpha}}$$

for $s \in B^{(y_0)}_1 := \{s \in B_1(0) : y_0 + ds \in \overline{\mathcal{N}^+_{r_s}} \setminus \Gamma_{w,r_s}\}$. By [GT, Theorem 6.29] and Lemma 3.8, for each $y_0 \in \overline{\mathcal{N}^+_{r_s}} \setminus \Gamma_{w,r_s}$, $w^{(y_0)}$ satisfies

$$\begin{aligned} |w^{(y_0)}|_{2,\alpha,B^{(y_0)}_{1/2}} &\leq C(|w|_{1,\alpha,\mathcal{N}^+_{r_s}} + \|\tilde{F}\|^{(-\alpha,\Gamma_w)}_{1,\alpha,\mathcal{N}^+_{r_s}} + \|\tilde{g}_1\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda} + \|\tilde{g}_3\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda}) \\ &\leq C(|w|_{0,\mathcal{N}^+_{r_s}} + \varsigma_1 + \varsigma_2 + \varsigma_4) \end{aligned} \tag{3.54}$$

where the second inequality above holds true by (3.11), (3.34) and (3.40).

By (3.39) and Lemma 2.5, reducing σ^* if necessary, we can have $\mu_f \geq \frac{1}{2}\mu_0 > 0$ on $\overline{\Lambda}$. Then, by (3.54), Remark 2.6 and [LI1, Corollary 2.5], we conclude that if w is a solution to (3.35) then it satisfies (3.36).

Step 4. Finally, we prove the existence of a solution to (3.45). Consider the following auxiliary problem:

$$\begin{cases} \partial_k(a_{jk}(y,0)\partial_j u) - u = \partial_k F_k & \text{in } \mathcal{N}^+_{r_s} \\ (a_{jk}(y,0)\partial_j u)\cdot\nu_s = g_1 & \text{on } S_{r_s} \\ (a_{jk}(y,0)\partial_j u)\cdot\nu_w = g_2 & \text{on } \Gamma_{w,r_s} \\ (a_{jk}(y,0)\partial_j u)\cdot\nu_{ex} = g_3 & \text{on } \Gamma_{ex} \end{cases} \tag{3.55}$$

for $F \in C^{\alpha}(\overline{\mathcal{N}^+_{r_s}})$, $g_1, g_3 \in C^{\alpha}(\overline{\Lambda})$ and $g_2 \in C^{\alpha}(\overline{\Gamma_{w,r_s}})$. (3.55) has a unique weak solution because the functional

$$I[u] = \frac{1}{2}\int_{\mathcal{N}^+_{r_s}} a_{jk}(y,0)\partial_j u \partial_k u + u^2 - 2F\cdot Du + \int_{\partial\mathcal{N}^+_{r_s}} (F\cdot\nu_{out} + g)u$$

has a minimizer over $W^{1,2}(\mathcal{N}^+_{r_s})$ with the outward unit normal ν_{out} of $\partial\mathcal{N}^+_{r_s}$ where we write

$$g := g_1\chi_{S_0} + g_2\chi_{\Gamma_{w,r_s}} - g_3\chi_{\Gamma_{ex}}$$

with $\chi_{\mathcal{D}}$ defined by

$$\chi_{\mathcal{D}}(y) = 1 \text{ for } y \in \mathcal{D}, \quad \chi_{\mathcal{D}}(y) = 0 \text{ otherwise} \quad \text{for } \mathcal{D} \subset \mathbb{R}^n.$$

For $F \in C^{1,\alpha}_{(-\alpha,\Gamma_w)}(\mathcal{N}^+_{r_s})$, $g_1, g_3 \in C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)$, the weak solution u also satisfies (3.36). By the method of continuity, we conclude that the linear boundary problem (3.35) has a solution $u \in C^{2,\alpha}_{(-1-\alpha,\Gamma_w)}(\mathcal{N}^+_f)$. By the comparison principle and [LI1, Corollary 2.5], the solution to (3.35) is unique. □

Remark 3.10. *According to the definition of the laplacian $\triangle_{S^{n-1}}$ on a sphere $\mathbb{S}^{n-1}$, we note*

$$\frac{1}{\omega^1}\partial_{y'_m}(b_{lm}\partial_{y'_l}u) = \triangle_{S^{n-1}} u$$

where we replace $-\triangle_{S^{n-1}}$ by $\triangle_{S^{n-1}}$ in the widely used convention(e.g. [BR]*) of $\triangle_{S^{n-1}}$.*

Lemma 3.4 is a direct result from Lemma 3.5.

Proof of Lemma 3.4. For $M, C^* > 0$, let σ^* be as in Lemma 3.5. Fix $(\Psi, \rho_-, \varphi_-, p_-, v_{ex}) \in \mathcal{B}^{(1)}_{\sigma_*}(Id, \rho_0^-, \varphi_0^-, p_0^-, v_c)$ and $f \in \mathcal{B}^{(2)}_{C^*\sigma^*}(r_s)$, and define a nonlinear mapping $\mathcal{G}_f$ by

$$\mathcal{G}_f : \phi \in \mathcal{K}_f(M) \mapsto u^{(\phi)} \in C^{2,\alpha}_{(-1-\alpha,\Gamma_w)}(\mathcal{N}^+_f)$$

where $u^{(\phi)}$ is a unique solution to (3.35) satisfying (3.36). By Lemma 3.5, if we choose M as

$$M = 6C \tag{3.56}$$

for the constant C in (3.36), then $\mathcal{G}_f$ maps $\mathcal{K}_f(M)$ into itself which is a Banach space.

For $k = 1, 2$, let us set $w_k = \mathcal{G}_f(\phi_k)$. By subtracting the boundary problem (3.45) for w_2 from the boundary problem (3.45) for w_1, one can directly show that there is a constant $\tilde{C}$ depending on the data so that there holds

$$\|\mathcal{G}_f(\phi_1) - \mathcal{G}_f(\phi_2)\|^{(-1-\alpha,\Gamma_w)}_{2,\alpha,\mathcal{N}^+_f} \le \tilde{C}\sigma^*\|\phi_1 - \phi_2\|^{(-1-\alpha,\Gamma_w)}_{2,\alpha,\mathcal{N}^+_f}. \tag{3.57}$$

So if we reduce $\sigma_* > 0$ satisfying Lemma 3.6, (3.42) ,(3.43) and $\tilde{C}\sigma^* < 1$, then the contraction mapping principle applies to $\mathcal{G}_f$. Therefore $\mathcal{G}_f$ has a unique fixed point, say, ψ. Obviously, ψ is a solution to (3.33) and it also satisfies Lemma 3.4(ii). From this, we choose $\sigma^\sharp = \sigma^*$. We note that the choice of $\sigma^\sharp$ depends on the data(in sense of Remark 2.7) and C^* . □

By Lemma 3.4, $\mathfrak{J}$, defined in (3.31), is well-defined when we replace σ^* by $\sigma^\sharp$ in (3.31).

For any given $(\Psi, \varphi_-, p_-, v_{ex}) \in \mathcal{B}^{(1)}_{\sigma^\sharp}(Id, \varphi_0^-, p_0^-, v_c)$ and $f \in \mathcal{B}^{(2)}_{C^*\sigma^\sharp}(r_s)$, let ψ^f be the fixed point of $\mathcal{G}_f$. Suppose that $f^* \in \mathcal{B}^{(2)}_{C^*\sigma^\sharp}(r_s)$ satisfies

$$\mathfrak{J}(\Psi, \varphi_-, p_-, v_{ex}, f^*) = (\varphi_- - \varphi_0^+ - \psi^{f^*})(f^*(x'), x') = 0 \text{ in } \Lambda.$$

Then

$$\varphi^{f^*} := \begin{cases} \varphi_- & \text{in } \overline{\mathcal{N}^-_{f^*}} \\ \varphi_0^+ + \psi^{f^*} & \text{in } \mathcal{N}^+_{f^*} \end{cases} \tag{3.58}$$

is a transonic shock solution of Problem 2 with a shock $S = \{r = f^*(x'), x'\} \cap \mathcal{N}$. Moreover, φ^{f^*} satisfies Proposition 3.1(a)-(c). Therefore, if there is a mapping $\mathfrak{S} : q = (\Psi, \varphi_-, p_-, v_{ex}) \mapsto f$ so that $\mathfrak{J}(q, \mathfrak{S}(q)) = 0$ holds, then Proposition 3.1 is proven by Lemma 3.4.

Lemma 3.11. *Fix $\alpha \in (0,1)$ and $C^* > 0$. For $\mathfrak{J}$ defined in* (3.31)*, there holds*

$$\mathfrak{J}(Id, \varphi_0^-, p_0^-, v_c, r_s) = 0. \tag{3.59}$$

Also, there exists a constant $\delta_* > 0$ depending on the data and C^* so that

(i) *$\mathfrak{J}$ is continuously Fréchet differentiable in $\mathcal{B}^{(1)}_{\delta_*}(Id, \varphi_0^-, p_0^-, v_c) \times \mathcal{B}^{(2)}_{C^*\delta_*}(r_s)$,*

(ii) *$D_f\mathfrak{J}(Id, \varphi_0^-, p_0^-, v_c, r_s) : C^{2,\alpha}_{(-1-\alpha,\partial\Lambda)}(\Lambda) \to C^{2,\alpha}_{(-1-\alpha,\partial\Lambda)}(\Lambda)$ is invertible.*

Proof. (3.59) is obvious.

Fix $q_0 = (\Psi, \varphi_-, p_-, v_{ex}) \in \mathcal{B}^{(1)}_{\sigma^\sharp/2}(Id, \varphi_0^-, p_0^-, v_c), f \in \mathcal{B}^{(2)}_{C^*\sigma^\sharp/2}(r_s),\ \tilde{q} = (\tilde{\Psi}, \tilde{\varphi}_-, \tilde{p}_-, \tilde{v}_{ex}) \in C^{2,\alpha}(\overline{\mathcal{N}}, \mathbb{R}^n) \times C^{3,\alpha}(\overline{\mathcal{N}^-_{r_s+\delta}}) \times C^{2,\alpha}(\overline{\mathcal{N}^-_{r_s+\delta}}) \times C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda), \tilde{f} \in C^{2,\alpha}_{(-1-\alpha,\partial\Lambda)}(\Lambda)$ with

$$\begin{aligned} &\|\tilde{q}\|_1 := \|\tilde{\Psi}\|_{2,\alpha,\mathcal{N}} + \|\tilde{\varphi}_-\|_{3,\alpha,\mathcal{N}^-_{r_s+\delta}} + \|\tilde{p}_-\|_{2,\alpha,\mathcal{N}^-_{r_s+\delta}} + \|\tilde{v}_{ex}\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda} = 1,\\ &\|\tilde{f}\|_2 := \|\tilde{f}\|^{(-1-\alpha,\partial\Lambda)}_{2,\alpha,\Lambda} = 1. \end{aligned} \tag{3.60}$$

To prove the Fréchet differentiability of $\mathfrak{J}$, we compute

$$\frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}\mathfrak{J}(q_0 + \varepsilon\tilde{q}, f + \varepsilon\tilde{f}) = \frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}(\varphi_- + \varepsilon\tilde{\varphi}_- - \varphi_0^+ - \psi_\varepsilon)(f(x') + \varepsilon\tilde{f}(x'), x') \tag{3.61}$$

where ψ_ε satisfies (3.35) in $\mathcal{N}^+_{f+\varepsilon\tilde{f}}$ with replacing $f, \phi, \Psi, p_-, v_{ex}$ by $f + \varepsilon\tilde{f}, \psi_\varepsilon, \Psi + \varepsilon\tilde{\Psi}, p_- + \varepsilon\tilde{p}_-, v_{ex} + \varepsilon\tilde{v}_{ex}$ respectively. Since the case of $(\tilde{\rho}_-, \tilde{\varphi}_-, \tilde{p}_-, \tilde{v}_{ex}) \neq (0,0,0,0)$ can be handled similarly, we assume $(\tilde{\rho}_-, \tilde{\varphi}_-, \tilde{p}_-, \tilde{v}_{ex}) = (0,0,0,0)$ for simplification.

To compute $\frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}\psi_\varepsilon(f(x') + \varepsilon\tilde{f}(x'), x')$, we use $\tilde{\Phi}_{f+\varepsilon\tilde{f}}$, given by

$$\tilde{\Phi}_{f+\varepsilon\tilde{f}} : (r, x') \mapsto (\frac{r_1 - f}{r_1 - f - \varepsilon\tilde{f}}(r - f(x') - \varepsilon\tilde{f}(x')) + f(x'), x'), \tag{3.62}$$

so that $\tilde{\psi}_\varepsilon := \psi_\varepsilon \circ (\Phi_{f+\varepsilon\tilde{f}})^{-1}$ is defined in $\mathcal{N}^+_f$ for every ε with $|\varepsilon| < \frac{\sigma^\sharp}{2}$. The corresponding boundary problem for $\tilde{\psi}_\varepsilon$ is as in (3.45) with replacing $f, \phi, \Psi, p_-, v_{ex}$ by $f+\varepsilon\tilde{f}, \psi_\varepsilon, \Psi+\varepsilon\tilde{\Psi}, p_-+\varepsilon\tilde{p}_-, v_{ex} + \varepsilon\tilde{v}_{ex}$ respectively.

$a_{jk}(x,\eta), F(x,m,\eta),\ b_1(x,\xi,\eta),\ g_1(x,m,p,\xi,\eta),\ g_2(x,m,\eta),\ g_3(x,z)$, defined in (3.6), (3.7), (3.28) and (3.34), are smooth with respect to their variables. So, subtraction the boundary problem for ψ_0 from the boundary problem for $\tilde{\psi}_\varepsilon$, one can show, by the elliptic estimates, that $\frac{\tilde{\psi}_\varepsilon - \psi_0}{\varepsilon}$ converges in $C^{2,\alpha}_{(-1-\alpha,\Gamma_w)}(\mathcal{N}^+_f)$ to u where u solves the following linear boundary problem

$$\begin{aligned} &\partial_k((a_{jk}(x, D\psi^0) + o^{(1)}_{jk})\partial_j u) = \partial_k(a_1 D\tilde{f} + a_2 D\tilde{\Psi}) \ \text{ in } \ \mathcal{N}^+_f,\\ &(b_1(D(\varphi^- - \varphi_0^-), D\psi^0) + o^{(2)}) \cdot Du - \mu_f u = a_3\tilde{f} + a_4 D\tilde{f} + a_5 D\tilde{\Psi} \ \text{ on } \ S_f,\\ &((a_{jk}(x, D\psi^0) + o^{(3)}_{jk})\partial_j u) \cdot \nu_w = a_6 D\tilde{f} + a_7 D\tilde{\Psi} \ \text{ on } \ \Gamma_{w,f},\\ &((a_{jk}(x, D\psi^0) + o^{(4)}_{jk})\partial_j u) \cdot \nu_{ex} = \tilde{v}_{ex} + a_8 D\tilde{f} + a_9 D\tilde{\Psi} \ \text{ on } \ \Gamma_{ex}. \end{aligned} \tag{3.63}$$

where $o^{(l)}_{jk}(l = 1, \cdots, 4), a_i(i = 1, \cdots, 9)$(whose dimensions can be understood from (3.63) so we only give symbolical expressions on the righthand sides of (3.63)) are (vector valued)

functions with the dependence of

$$o_{jk}^{(l)} = o_{jk}^{(l)}(x, D\Psi, D\psi^0),$$
$$a_i = a_i(D\Psi, (\varphi_-, p_-) - (\varphi_0^-, p_0^-), D(\varphi_- - \varphi_0^-), D\psi_0, f, Df),$$

and, by Lemma 3.4, each $o_{jk}^{(l)}$ satisfies

$$\|o_{jk}^{(l)}(x, D\Psi, D\psi^0)\|_{1,\alpha,\mathcal{N}_f^+}^{(-\alpha,\Gamma_w)} \le C\|\psi^0\|_{2,\alpha,\mathcal{N}_f^+}^{(-1-\alpha,\Gamma_w)} \le C\sigma^\sharp. \tag{3.64}$$

Moreover, it is not hard to check that $o_{jk}^{(l)}$, a_i are smooth with respect to their arguments in a small neighborhood of the background state. By the smoothness of $o_{jk}^{(l)}, a_i$, and the smallness of $o_{jk}^{(l)}$ obtained from (3.64) reducing $\sigma^\sharp$ if necessary, (3.63) has a unique solution $u \in C^2(\mathcal{N}_f^+) \cap C^0(\overline{\mathcal{N}}_f^+)$.

By subtracting the boundary problem for u from the boundary problem for $\frac{\tilde{\psi}_\varepsilon - \psi_0}{\varepsilon}$, one can also check that there exists a constant C so that there holds $\|\frac{1}{\varepsilon}(\tilde{\psi}^\varepsilon - \psi^0) - u\|_{2,\alpha,\mathcal{N}_f^+}^{(-1-\alpha,\Gamma_w)} \le C\varepsilon$, or equivalently

$$\|\tilde{\psi}^\varepsilon - \psi^0 - \varepsilon u\|_{2,\alpha,\mathcal{N}_f^+}^{(-1-\alpha,\Gamma_w)} \le C\varepsilon^2. \tag{3.65}$$

We emphasize that the constant C in (3.65) is independent of the choice of $\tilde{q}$ and $\tilde{f}$, but only depends on the data in sense of Remark 2.7.

In addition, $o_{jk}^{(l)}, a_i$ continuously depend on their variables in the corresponding norm for each component, also the modulus of the continuity is uniform over $\mathcal{B}_{\sigma^\sharp}^{(1)}(Id, \varphi_0^-, p_0^-, v_c) \times \mathcal{B}_{C^*\sigma^\sharp}^{(2)}(r_s)$.

Back to (3.61), we have shown

$$\begin{aligned} &\frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}\mathfrak{I}(q_0 + \varepsilon\tilde{q}, f + \varepsilon\tilde{f})(x') \\ &= \tilde{f}(x')\partial_r(\varphi_- - \varphi_0^+ - \psi_0)(f(x'), x') + (\tilde{\varphi}_- - u)(f(x'), x'). \end{aligned} \tag{3.66}$$

By (3.63), u linearly depends on $\tilde{q}$ and $\tilde{f}$. So the mapping $\mathcal{L}$ defined by

$$\mathcal{L} : (\tilde{q}, \tilde{f}) \mapsto \frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}\mathfrak{I}(q_0 + \varepsilon\tilde{q}, f + \varepsilon\tilde{f})$$

is a bounded linear mapping from $C^{2,\alpha}(\overline{\mathcal{N}}, \mathbb{R}^n) \times C^{3,\alpha}(\overline{\mathcal{N}_{r_s+\delta}^-}) \times C^{2,\alpha}(\overline{\mathcal{N}_{r_s+\delta}^-}) \times C_{(-\alpha,\partial\Lambda)}^{1,\alpha}(\Lambda) \times C_{(-1-\alpha,\partial\Lambda)}^{2,\alpha}(\Lambda)$ to $C_{(-1-\alpha,\partial\Lambda)}^{2,\alpha}(\Lambda)$ where we understand $C_{(-1-\alpha,\partial\Lambda)}^{2,\alpha}(\Lambda)$ in sense of Remark 2.6.

It is easy to see from (3.66), (3.65) that, for $(q, f) \in \mathcal{B}_{\sigma^\sharp}^{(1)}(Id, \rho_0^-, \varphi_0^-, p_0^-, v_c) \times \mathcal{B}_{C^*\sigma^\sharp}^{(2)}(r_s)$, $D\mathfrak{I}(q, f)$ is given by

$$D\mathfrak{I}(q, f) : (\tilde{q}, \tilde{f}) \mapsto \frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}\mathfrak{I}(q_0 + \varepsilon\tilde{q}, f + \varepsilon\tilde{f}), \tag{3.67}$$

and $D\mathfrak{I}$ is continuous in (q, f). This verifies Lemma 3.11(i).

Next, we prove Lemma 3.11(ii). By (3.7), (3.28), (3.34), we have, for any ψ,

$$F_k(x, Id, D\psi) = g_1(x, Id, 0, D\psi) = g_2(x, Id, D\psi) = g_3(v_c) = 0 \tag{3.68}$$

for $k = 1, \cdots, n$. Lemma 3.4, (3.61) and (3.68) provide

$$\frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}\mathfrak{I}(Id, \varphi_0^-, p_0^-, v_c, r_s + \varepsilon\tilde{f}) = \frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}(\varphi_0^- - \varphi_0^+)(r_s + \varepsilon\tilde{f}) = \tilde{f}\partial_r(\varphi_0^- - \varphi_0^+)(r_s).$$

Thus, we obtain

$$D_f\mathfrak{J}(Id, \varphi_0^-, p_0^-, v_c, r_s) : \tilde{f} \mapsto \tilde{f}\partial_r(\varphi_0^- - \varphi_0^+)(r_s) \tag{3.69}$$

By (2.32) and (2.35), we have $\partial_r(\varphi_0^- - \varphi_0^+)(r_s) > 0$. Thus we conclude that $D_f\mathfrak{J}$ is invertible at $(Id, \varphi_0^-, p_0^-, v_c, r_s)$. Here, we choose $\delta = \frac{\sigma^\sharp}{2}$ where $\sigma^\sharp > 0$ depends on the data and C^*. □

Proof of Proposition 3.1. Let us fix $C^* = 10$ then the choice of σ^* is fixed accordingly, and moreover the choice of σ_* only depends on the data in sense Remarkn 2.7. By Lemma 3.11 and the implicit mapping theorem, there exists a constant $0 < \sigma_3 < \delta_*$ depending on the data in sense of Remark 2.7 so that there is a unique mapping $\mathcal{S} : \mathcal{B}^{(1)}_{\sigma_3}(Id, \varphi_0^-, p_0^-, v_c) \to \mathcal{B}^{(2)}_{C^*\sigma^*}(r_s)$ satisfying

$$\mathfrak{J}(q, \mathcal{S}(q)) = 0 \quad \text{for all} \quad q \in \mathcal{B}^{(1)}_{\sigma_3}(Id, \varphi_0^-, p_0^-, v_c), \tag{3.70}$$

and $\mathcal{S}$ is continuously Fréchet differentiable in $\mathcal{B}^{(1)}_{\sigma_3}(Id, \varphi_0^-, p_0^-, v_c)$.

By (2.19), Lemma 3.4, and the observation made after the proof of Lemma 3.4, (3.70) implies that $f^{(q)} = \mathcal{S}(q)$ and $\varphi^{f^{(q)}}$ in (3.58) satisfy Proposition 3.1 (a)-(c). The uniqueness of a solution of Problem 3 also follows from the implicit mapping theorem. We note that the choice of $\sigma_3 > 0$ only depends on the data because σ_3 depends on the data and δ_* which depends on the data and C^* but we fixed C^* as $C^* = 10$. □

4. Transport Equation for Pressure p

This section is devoted to step 2 in section 2.6.

4.1. **The method of characteristics.** By the choice of σ_3 in Proposition 3.1, if φ is a solution to Problem 3, then $\partial_r\varphi$ is bounded below by a positive constant in $\mathcal{N}^+$. So r can be regarded as a time-like variable and then we apply the method of characteristics to solve (2.43) for p. According to (3.2), however, $D\varphi$ is not Lipschitz continuous up to $\partial\mathcal{N}^+$. So the unique solvability of (2.43) needs to be established first, and we will prove the following proposition.

Proposition 4.1. *Let σ_3 be as in Proposition 3.1. For any given $\alpha \in (0,1)$ there exist constant $C > 0$ and $\sigma' \in (0, \sigma_3]$ depending on the data in sense of Remark 2.7 so that if $\varsigma_1 + \varsigma_2 + \varsigma_4 \le \sigma'$, and φ is a solution of Problem 3 with a shock $S = \{r = f(x')\}$, then* (2.43) *with the initial condition* (2.49) *on S has a unique solution p satisfying*

$$\|p - p_0^+\|^{(-\alpha, \Gamma_w)}_{1,\alpha,\mathcal{N}^+} \le C(\varsigma_1 + \varsigma_2 + \varsigma_4). \tag{4.1}$$

We denote as $\mathcal{N}^+$ for the subsonic domain $\{|D\Psi^{-1}D\varphi| < c\} = \{r > f(x')\}$. We first note that φ is in C^2 up to the wall Γ_w of $\mathcal{N}^+$ away from the corners.

Lemma 4.2. *Given $(\Psi, \varphi_-, p_-, v_{ex}) \in \mathcal{B}^{(1)}_{\sigma_3}(Id, \varphi_0^-, p_0^-, v_c)$ satisfying Proposition 3.1 (i), (ii) reducing σ_3 if necessary, φ, the corresponding solution of Problem 3, with a shock S satisfies*

$$\|\varphi - \varphi_0^+\|^{(-1-\alpha, \overline{S}\cup\overline{\Gamma_{ex}})}_{2,\alpha,\mathcal{N}^+} \le C(\varsigma_1 + \varsigma_2 + \varsigma_4) \tag{4.2}$$

for $\varsigma_1, \varsigma_2, \varsigma_4$ in Theorem 1 and Proposition 3.1.

To prove Lemma 4.1, it suffices to show that the nonlinear boundary problem (3.33) in the fixed domain $\mathcal{N}^+$ has a unique solution ψ so that $\varphi = \varphi_0^+ + \psi$ satisfies (4.2), and this can be proven by a similar method to the proof of Lemma 3.4 except that the scaling argument in step 4 needs to be performed for any point $y_0 \in \mathcal{N}^+ \setminus (\overline{S} \cup \overline{\Gamma_{ex}})$. So we skip the proof of Lemma 4.1.

Considering r as a time-like variable, it is more convenient to use a changes of variables to transform $\mathcal{N}^+$ to $\mathcal{N}^+_{r_s}$ keeping (4.2) in the transformed domain $\mathcal{N}^+_{r_s}$. This requires a new change of variables:

$$\begin{aligned} G_\varphi(r,x') &= \big((k(\varphi_- - \varphi)(r,x') + r_s)(1-\chi(r)) + r\chi(r), x'\big) \\ &=: (v^{(\varphi)}(r,x'), x') =: (\tilde{r}, x') \end{aligned} \tag{4.3}$$

with a smooth function $\chi(r)$ satisfying $\chi(r) = \begin{cases} 0 & \text{if } r \le r_s + \frac{r_1 - r_s}{10} \\ 1 & \text{if } r \ge r_1 - \frac{r_1 - r_s}{2} \end{cases}$ and $\chi'(r) \ge 0$ for $r > 0$. By (4.2), if $\varsigma_1 + \varsigma_2 + \varsigma_4 = \sigma$ is small depending on the data in sense of Remark 2.7, and we choose a constant k satisfying

$$\begin{aligned} &C\sigma \le \frac{1}{10}\min\{(\varphi_0^- - \varphi_0^+)(r_1), \partial_r(\varphi_0^- - \varphi_0^+)(r_s)\}, \\ &k = \frac{r_1 - r_s}{8(\varphi_0^- - \varphi_0^+)(r_1)}, \end{aligned} \tag{4.4}$$

then we have

$$\partial_r v^{(\varphi)} \ge \frac{r_1 - r_s}{4} > 0 \quad \text{in} \quad \mathcal{N}^+_{r_s - 2\delta}, \tag{4.5}$$

thus G_φ is invertible.

We choose x' in (2.22) so that $(\tilde{r}, x')$ forms an orthogonal coordinate system. To specify this choice, we write (r,x') as (r,ϑ) with $\vartheta = (\vartheta_1, \cdots, \vartheta_{n-1})$ for the rest of paper. And then, we apply the change of variables (4.3) to rewrite (2.43) in $\mathcal{N}^+$ as

$$V\cdot DE = (V\cdot\hat{\tilde{r}})\partial_{\tilde{r}}E + \sum_{j=1}^{n-1}(V\cdot\hat{\vartheta}_j)\partial_{\vartheta_j}E = 0 \ \text{ for } \ (\tilde{r},\vartheta)\in(r_s,r_1]\times\Lambda \tag{4.6}$$

with

$$V = J_\varphi(D\Psi^{-1})^T(D\Psi^{-1})D\varphi, \quad E\circ G_\varphi = \frac{p}{(B_0 - \frac{1}{2}|D\Psi^{-1}D\varphi|^2)^{\frac{\gamma}{\gamma-1}}} \tag{4.7}$$

where J_φ is the Jacobian matrix associated with the change of variables in (4.3). Here, $\hat{\tilde{r}}, \hat{\vartheta}_j$ are unit vectors pointing the positive directions of $\tilde{r}$ and ϑ_j respectively. By Proposition 3.1, Lemma 4.2 and (4.3), we have

Lemma 4.3. *Set $V_0 := J_{\varphi_0^+}D\varphi_0^+$. Then, V defined in (4.7) satisfies*

$$\|V - V_0\|^{(-\alpha,\Gamma_w)}_{1,\alpha,\mathcal{N}^+_{r_s}} + \|V - V_0\|^{(-\alpha,\overline{S_0}\cup\overline{\Gamma_{ex}})}_{1,\alpha,\mathcal{N}^+_{r_s}} \le C(\varsigma_1+\varsigma_2+\varsigma_4)$$

for $\varsigma_1,\varsigma_2,\varsigma_4$ in Theorem 1 and Proposition 3.1. Thus, reducing σ_3 in Proposition 3.1 if necessary, there is a constant $\omega_0 > 0$ depending on the data in sense of Remark 2.7 so that there holds $V\cdot\hat{\tilde{r}} \ge \omega_0 > 0$ in $\overline{\mathcal{N}^+_{r_s}}$.

The proof is obtained by a straightforward computation so we skip here.

Lemma 4.4. *Let us set $E_0^+ := \frac{p_0^+}{(B_0 - \frac{1}{2}|\partial_r\varphi_0^+|^2)^{\frac{\gamma}{\gamma-1}}}$, then E_0^+ is a constant.*

Proof. For V_0 defined in Lemma 4.3, by (4.8), E_0^+ satisfies

$$V_0\cdot DE_0^+ = (V_0\cdot\hat{r})\partial_r E_0^+ = 0.$$

Since $V_0 \cdot \hat{r} > 0$ for all $r \in [r_s, r_1]$, and $E_0^+(r_s)$ is a constant by (2.32), (2.33), therefore we conclude $E_0^+(r) = E_0^+(r_s) = constant$ for all $r \in [r_s, r_1]$. □

By Lemma 4.3, we can divide the equation (4.6) by $V \cdot \hat{\tilde{r}}$ so that the radial speed of all characteristics associated with V is 1. From now on, we consider the equation

$$\partial_{\tilde{r}} E + \sum_{j=1}^{n-1} \frac{V \cdot \hat{\vartheta}_j}{V \cdot \hat{\tilde{r}}} \partial_{\vartheta_j} E = 0 \ \text{ in } \ \mathcal{N}_{r_s}^+. \tag{4.8}$$

We denote

$$W^* = (1, \frac{V \cdot \hat{\vartheta}_1}{V \cdot \hat{\tilde{r}}}, \cdots, \frac{V \cdot \hat{\vartheta}_{n-1}}{V \cdot \hat{\tilde{r}}}). \tag{4.9}$$

The initial condition for E is given from (2.49). Setting $Q_2 := (D\Psi^{-1})^T D\Psi^{-1}$, $E(r, \vartheta)$ must satisfy $E(r_s, \vartheta) = E_{int}(\vartheta)$ for all $\vartheta \in \Lambda$ where we set

$$E_{int}(\vartheta) := \frac{\rho_-(\frac{|D(\varphi_- - \varphi)|_{Q_2}}{|D\Psi^{-1} D(\varphi_- - \varphi)|} D\varphi_- \cdot \nu_s)^2 + p_- - \rho_- K_s}{(B_0 - \frac{1}{2}|D\Psi^{-1}(\Psi) D\varphi|^2)^{\frac{\gamma}{\gamma-1}}} \Big|_{(r,\vartheta)=(f(\vartheta),\vartheta)} . \tag{4.10}$$

Above, $f(\vartheta)$ is the location of the transonic shock for the potential φ in Proposition 3.1. To simplify notation, we write $\tilde{r}$ as r hereafter. By (4.8) and (4.10), Proposition 4.1 is a direct result of the following lemma.

Lemma 4.5. *Fix $\alpha \in (0, 1)$, and suppose that a vector valued function $W = (1, W_2, \cdots, W_n)$ satisfies $W \cdot \nu_w = 0$ on $\Gamma_{w,r_s} = \partial \mathcal{N}_{r_s}^+ \cap \Gamma_w$ for a unit normal ν_w on Γ_w^+, and*

$$\|W\|_{1,\alpha,\mathcal{N}_{r_s}^+}^{(-\alpha,\Gamma_w)} + \|W\|_{1,\alpha,\mathcal{N}_{r_s}^+}^{(-\alpha,\overline{S_{r_s}} \cup \overline{\Gamma_{ex}})} \le k \tag{4.11}$$

for some constant $k > 0$. Then the initial value problem

$$\partial_r F + \sum_{j=1}^{n-1} W_{j+1} \partial_{\vartheta_j} F = 0 \ \textit{ in } \ \mathcal{N}_{r_s}^+, \quad F(r_s, \vartheta) = F_0(\vartheta) \ \textit{ for } \ \vartheta \in \Lambda \tag{4.12}$$

has a unique solution F satisfying

$$\|F\|_{1,\alpha,\mathcal{N}_{r_s}^+}^{(-\alpha,\Gamma_w)} \le C \|F_0\|_{1,\alpha,\Lambda}^{(-\alpha,\partial\Lambda)} \tag{4.13}$$

for a constant $C > 0$ depending on $n, \Lambda, r_s, r_1, \alpha$ and k.

Proof of Proposition 4.1.
By (2.53) and Lemma 4.3, $W^* = (1, \frac{V \cdot \hat{\vartheta}_1}{V \cdot \hat{r}}, \cdots, \frac{V \cdot \hat{\vartheta}_{n-1}}{V \cdot \hat{r}})$ in (4.8) satisfies all the assumptions of Lemma 4.5. So, (4.8) with (4.10) has a unique solution E. Then the pressure function p is given by

$$p = (B_0 - \frac{1}{2}|D\Psi^{-1} D\varphi|^2)^{\frac{\gamma}{\gamma-1}} \cdot (E \circ G_\varphi).$$

It remains to verify (4.1). By Lemma 4.4, if E satisfies (4.8) then so does $E - E_0^+$. Then, by (4.13), we obtain

$$\|E - E_0^+\|_{1,\alpha,\mathcal{N}_{r_s}^+}^{(-\alpha,\Gamma_w)} \le C \|E_{int} - E_0^+\|_{1,\alpha,\Lambda}^{(-\alpha,\partial\Lambda)}, \tag{4.14}$$

where E_{int} is given by (4.10).

Moreover, by (4.10), one can explicitly show

$$\|E_{int} - E_0^+\|_{1,\alpha,\Lambda}^{(-\alpha,\partial\Lambda)} \le C(\varsigma_1 + \varsigma_2 + \varsigma_4). \tag{4.15}$$

This with (3.2) gives (4.1). We choose $\sigma' = \sigma_3$ by reducing σ_3 if necessary to satisfy (4.2) and (4.4). □

Now it remains to prove Lemma 4.5. Consider the following ODE system

$$\begin{aligned} &\dot{X}(t) = -W(X(t)) \ \text{ for } \ r < t \le 2r - r_s \\ &X(r) = (r, \vartheta) \end{aligned} \tag{4.16}$$

for $(r,\vartheta) \in \mathcal{N}_{r_s}^+ \cup \Gamma_{ex}$. The slip boundary condition $W \cdot \nu_w = 0$ on Γ_w^+ implies that any characteristic curve X does not penetrate out of $\mathcal{N}_{r_s}^+$. If (4.16) has a unique solution $X(t;r,\vartheta)$ for any given $(r,\vartheta) \in \mathcal{N}_{r_s}^+ \cup \Gamma_{ex}$, then we can define a mapping $\mathcal{J} : \mathcal{N}_{r_s}^+ \cup \Gamma_{ex} \to \Lambda$ by

$$\mathcal{J} : (r,\vartheta) \mapsto X(2r - r_s; r, \vartheta). \tag{4.17}$$

If $\mathcal{J}$ is differentiable then $F = F_0 \circ \mathcal{J}$ is a solution to (4.12).

4.2. **The system of ODEs** (4.16)**.** (4.16) has a solution for any given $(r,\vartheta) \in (r_s, r_1] \times \Lambda$ by the Cauchy-Peano theorem [AM, 7.3]. But the standard ODE theory is not sufficient to claim the uniqueness of a solution for (4.16) because W is only in $C^\alpha(\overline{\mathcal{N}_{r_s}^+})$. To prove the uniqueness, we will use the fact that DW is integrable along each solution X to (4.16). Denote as $X(t;r,\vartheta)$ for a solution to (4.16). Hereafter, we call each $X(t;r,\vartheta)$ a *characteristic* associated with W initiated from (r,ϑ).

Remark 4.6. *Writing $X = (X_1, \cdots, X_n)$, it is easy to check*

$$X_1(t; r, \vartheta) = 2r - t \tag{4.18}$$

for any $(r,\vartheta) \in (r_s, r_1] \times \Lambda$. So, we restrict (4.16) for $t \in [2r - r_1, 2r - r_s]$ because of $X_1(2r - r_1; r, \vartheta) = r_1$, $X_1(2r - r_s; r, \vartheta) = r_s$.

Lemma 4.7. *For any given $(r,\vartheta) \in (r_s, r_1] \times \Lambda$, (4.16) has a unique solution. Moreover there is a constant $C > 0$ depending on Λ, n and k satisfying*

$$\frac{1}{C} dist((r,\vartheta), \Gamma_w) \le dist(X(t;r,\vartheta), \Gamma_w) \le C dist((r,\vartheta), \Gamma_w) \tag{4.19}$$

for $t \in [2r - r_1, 2r - r_s]$.

Proof. Since W is in $C^\alpha(\overline{\mathcal{N}_{r_s}^+})$, (4.16) has a solution $X(t;r,\vartheta)$ for any $(r,\vartheta) \in (r_s, r_1] \times \Lambda$. Fix $r \in (r_s, r_1]$. Let $X^{(1)}$ and $X^{(2)}$ be solutions to (4.16) with

$$X^{(1)}(r) = (r, \vartheta_1), \quad X^{(2)}(r) = (r, \vartheta_2) \tag{4.20}$$

for $\vartheta_1, \vartheta_2 \in \Lambda$. By (4.16), we have

$$\begin{aligned} \frac{d}{dt}|X^{(1)} - X^{(2)}|^2 &= 2(W(X^{(2)}) - W(X^{(1)})) \cdot (X^{(1)} - X^{(2)}) \\ &\le C|DW(2r - t, \cdot)|_{L^\infty(\Lambda)} |X^{(1)} - X^{(2)}|^2. \end{aligned}$$

In $(r,\vartheta)-$coordinates, it is easy to see

$$dist((\tilde{r}, \tilde{\vartheta}), \overline{S_0} \cup \overline{\Gamma_{ex}}) = \min\{r_1 - \tilde{r}, \tilde{r} - r_s\} \quad \text{for} \ \ (\tilde{r}, \tilde{\vartheta}) \in \mathcal{N}_{r_s}^+.$$

So there is a constant C depending on $n, \mathcal{N}_{r_s}^+$ so that DW satisfies

$$|DW(2r-t,\vartheta)| \le Ck(1+(2r-r_s-t)^{-1+\alpha}+(r_1-2r+t)^{-1+\alpha}) =: m(t) \tag{4.21}$$

for any $\vartheta \in \Lambda$ and $t \in [r, 2r-r_s]$ where k is as in (4.11). Then we apply the method of the integrating factor to obtain

$$\frac{d}{dt}(e^{-\int_r^t m(t')dt'}|X^{(1)}-X^{(2)}|^2(t)) \le 0 \quad \text{for } t \in [r, 2r-r_s]. \tag{4.22}$$

If $\vartheta_1 = \vartheta_2$ then (4.20) and (4.22) imply $X^{(1)}(t) = X^{(2)}(t)$ for all $t \in [r, 2r-r_s]$. We have accomplished the unique solvability of (4.16).

Since $W\cdot\nu_w = 0$ on Γ_w^+, any characteristic, initiated from a point on the wall of $\mathcal{N}_{r_s}^+$, remains on the wall i.e., if $\vartheta_2 \in \partial\Lambda$, then $X^{(2)}$ lies on Γ_w. Then (4.22) implies

$$\begin{aligned} dist(X^{(1)}(t),\Gamma_w) &= \inf_{\vartheta_2\in\partial\Lambda}|X^{(1)}(t)-X^{(2)}(t)| \\ &\le C \inf_{\vartheta_2\in\partial\Lambda}|(r,\vartheta_1)-(r,\vartheta_2)| = Cdist((r,\vartheta_1),\Gamma_w). \end{aligned}$$

This verifies the second inequality of (4.19). One can argue similarly to verify the first inequality of (4.19). □

4.3. Regularity of $\mathcal{J} : (r,\vartheta) \mapsto X(2r-r_s; r, \vartheta)$ in (4.17).

Lemma 4.8. *Fix $\alpha \in (0,1)$, and let k be as in Lemma 4.5. The mapping $\mathcal{J}$ defined in (4.17) is continuously differentiable, and it satisfies*

$$\|\mathcal{J}\|_{1,\alpha,\mathcal{N}_{r_s}^+}^{(-1,\Gamma_w)} \le C$$

for a constant $C > 0$ depending on $n, \Lambda, r_s, r_1, \alpha$ and k.

Proof. First of all, let us assume that $\mathcal{J}$ satisfies

$$|\mathcal{J}|_{1,0,\mathcal{N}_{r_s}^+} \le C, \tag{4.23}$$

and we estimate $[D\mathcal{J}]_{\alpha,\mathcal{N}_{r_s}^+}^{(0,\Gamma_w)}$. If $\mathcal{J}$ is differentiable then (4.16) implies

$$\begin{aligned} &(a)\ \ \partial_r\mathcal{J}(r,\vartheta) = Z(2r-r_s;r,\vartheta) - 2W(X(2r-r_s;r,\vartheta)),\\ &(b)\ \ \partial_\vartheta\mathcal{J} = Z(2r-r_s;r,\vartheta) \end{aligned} \tag{4.24}$$

where $Z(t;r,\vartheta)$ is the solution to the ODE system

$$\begin{aligned} &\dot{Z}(t;r,\vartheta) = -DW(X(t;r,\vartheta))Z(t;r,\vartheta),\\ &Z(r;r,\vartheta) = \begin{cases} (1,0,\cdots,0)+W(r,\vartheta) & \text{for (a)}\\ (0,\delta_{1j},\cdots,\delta_{n-1,j}) & \text{for (b).}\end{cases} \end{aligned} \tag{4.25}$$

Fix $(r,\vartheta),(r',\vartheta') \in (r_s, r_1]\times\Lambda$ with $(r,\vartheta)\neq(r',\vartheta')$, and consider $J := \frac{|D\mathcal{J}(r,\vartheta)-D\mathcal{J}(r',\vartheta'))|}{|(r,\vartheta)-(r',\vartheta')|^\alpha}$. For simplicity, we assume $r_s < r < r' < r_1$ and $\vartheta = \vartheta'$. Under this assumption, we first estimate $\frac{|\mathcal{J}_r(r,\vartheta)-\mathcal{J}_r(r',\vartheta)|}{|r-r'|^\alpha}$ because the case of $\vartheta\neq\vartheta'$ can be handled similarly. Also, the estimate of $\frac{|\mathcal{J}_\vartheta(r,\vartheta)-\mathcal{J}_\vartheta(r',\vartheta')|}{|(r,\vartheta)-(r',\vartheta')|^\alpha}$ is similar and even simpler then the estimate of $\frac{|\mathcal{J}_r(r,\vartheta)-\mathcal{J}_r(r',\vartheta)|}{|r-r'|^\alpha}$.

By (4.11) and (4.23), it is easy to check

$$\frac{2|W(X(2r-r_s;r,\vartheta))-W(X(2r'-r_s;r',\vartheta))|}{|r-r'|^\alpha} \le C \tag{4.26}$$

for a constant C depending on $n, \Lambda, r_s, r_1, \alpha$ and k. For the rest of the proof, a constant C depends on $n, \Lambda, r_s, r_1, \alpha$ and k unless otherwise specified.

To estimate $\frac{|Z(2r-r_s;r,\vartheta)-Z(2r'-r_s;r',\vartheta)|}{|r-r'|^\alpha}$, we split the numerator as

$$\begin{aligned}&|Z(2r-r_s;r,\vartheta)-Z(2r'-r_s;r',\vartheta)|\\ &\le |Z(2r-r_s;r,\vartheta)-Z(2r-r_s;r',\vartheta)|+|Z(2r-r_s;r',\vartheta)-Z(2r'-r_s;r',\vartheta)|,\end{aligned}$$

and consider them separately.

By (4.21) and (4.25), one can easily check

$$|Z(2r-r_s;r',\vartheta)-Z(2r'-r_s;r',\vartheta)|\le C(r'-r)^\alpha. \tag{4.27}$$

Let us denote as $Z_1(t)$ and $Z_2(t)$ for $Z(t;r,\vartheta)$ and $Z(t;r',\vartheta)$ respectively. Then, by (4.25), $z:=|Z_1-Z_2|^2$ satisfies

$$\dot z\le C(|DW(X(t;r,\vartheta))|z+|DW(X(t;r,\vartheta))-DW(X(t;r',\vartheta))|z^{1/2}) \tag{4.28}$$

on the interval $I_{r,r'}:=[2r'-r_1,2r'-r_s]\cap[2r-r_1,2r-r_s]$. By (4.23), we have

$$\begin{aligned}&|DW(X(t;r,\vartheta))-DW(X(t;r',\vartheta))|\\ &\le C|r-r'|^\alpha\frac{|DW(X(t;r,\vartheta))-DW(X(t;r',\vartheta))|}{|X(t;r,\vartheta)-X(t;r',\vartheta)|^\alpha}.\end{aligned}$$

Set $d:=\min\{dist((r,\vartheta),\Gamma_w),dist((r',\vartheta),\Gamma_w)\}$ and

$$\mathcal{Z}(t):=\frac{|DW(X(t;r,\vartheta))-DW(X(t;r',\vartheta))|}{|X(t;r,\vartheta)-X(t;r',\vartheta)|^\alpha}.$$

For a constant $\beta\in(0,1)$, we estimate $\mathcal{Z}^\beta$ and $\mathcal{Z}^{1-\beta}$ using two different weighted Hölder norms of W. By (4.19), for any $t\in I_{r,r'}$, we have

$$\mathcal{Z}^\beta\le Cd^{-\beta}(\|W\|^{(-\alpha,\Gamma_w)}_{1,\alpha,\mathcal{N}^+_{r_s}})^\beta\le Ck^\beta d^{-\beta}. \tag{4.29}$$

Next, set $d_*(t):=\min\{dist(X(t;r,\vartheta),\overline{S_0}\cup\overline{\Gamma_{ex}}),dist(X(t;r',\vartheta),\overline{S_0}\cup\overline{\Gamma_{ex}})\}$. Then $d_*(t)$ can be expressed as

$$d_*(t)=\min\{2r-t-r_s,r_1-2r'+t\}\ \text{ for }\ t\in I_{r,r'}.$$

So $(d_*(t))^{-1+\beta}$ is bounded by $m_*(t)$ which is integrable over $I_{r,r'}$, then we have

$$\mathcal{Z}^{1-\beta}(t)\le Cm_*(t)(\|W\|^{(-\alpha,\overline{S_0}\cup\overline{\Gamma}_{ex})}_{1,\alpha,\mathcal{N}^+_{r_s}})^{1-\beta}\le Ck^{1-\beta}m_*(t). \tag{4.30}$$

Back to (4.21), by (4.22), (4.25), (4.29), (4.30) and the method of integrating factor, we obtain

$$\begin{aligned}&\frac{d}{dt}(e^{-\int_r^t m(t')dt'}z)\le(r'-r)^\alpha Ckd^{-\beta}m_*z^{1/2}\\ &\Rightarrow z(t)\le C((r'-r)^2+(r'-r)^\alpha d^{-\beta}\sup_{t\in I_{r,r'}}z^{1/2})\ \text{ for all } t\in I_{r,r'}\\ &\Rightarrow \sup_{t\in I_{r,r'}}z\le C((r'-r)^2+d^{-2\beta}(r'-r)^{2\alpha})\end{aligned}$$

which provides

$$\frac{|Z(2r-r_s;r,\vartheta)-Z(2r-r_s;r',\vartheta)|}{|r'-r|^\alpha}\le Cd^{-\beta} \tag{4.31}$$

for any $\beta \in (0,1)$ and a constant C depending on $n, \mathcal{N}_{r_s}^+, k$ and β. The case of $\vartheta \neq \vartheta'$ can be handled in the same fashion. The estimate of $\frac{|\partial_\vartheta \mathcal{J}(r,\vartheta) - \partial_\vartheta \mathcal{J}(r',\vartheta')|}{|(r,\vartheta)-(r',\vartheta')|^\alpha}$ is even simpler then the estimate of $\frac{|\partial_r \mathcal{J}(r,\vartheta) - \partial_r \mathcal{J}(r',\vartheta')|}{|(r,\vartheta)-(r',\vartheta')|^\alpha}$ so we skip the details and we conclude that, for any constant $\beta \in (0,1)$, there is a constant C so that there holds

$$[D\mathcal{J}]^{(\beta-\alpha,\Gamma_w)}_{\alpha,\mathcal{N}_{r_S}^+} \le C.$$

By means of similar arguments, one can directly prove $\mathcal{J} \in C^1(\overline{\mathcal{N}_{r_s}^+})$. □

Remark 4.9. *$\mathcal{J}$ is in $C^{1,\alpha}_{(-1-\alpha+\beta,\Gamma_w)}(\mathcal{N}_{r_s}^+)$ for any $\beta \in (0,1)$.*

Now, we are ready to prove Lemma 4.5.

Proof of Lemma 4.5. By Lemma 4.8, $F := F_0 \circ \mathcal{J}$ is a solution to (4.12). By (4.19), we easily get the estimate $\|F\|^{(-\alpha,\Gamma_w)}_{1,0,\mathcal{N}_{r_s}^+} \le C\|F_0\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda}$ so it is remains to estimate $[F]^{(0,\Gamma_w)}_{\alpha,\mathcal{N}_{r_s}^+}$. Fix (r,ϑ) and (r',ϑ') in $(r_s,r_1] \times \Lambda$, and assume $(r,\vartheta) \neq (r',\vartheta')$. Let us set

$$d := \frac{1}{2}\min\{dist((r,\vartheta),\Gamma_w), dist((r',\vartheta'),\Gamma_w)\} > 0.$$

Setting $[\zeta]_a^b := \zeta(b) - \zeta(a)$, we have

$$\begin{aligned}
&|DF(r,\vartheta) - DF(r',\vartheta')| \\
&\le |[D\mathcal{J}]^{(r,\vartheta)}_{(r',\vartheta')}||DF_0(\mathcal{J}(r,\vartheta))| + |D\mathcal{J}(r',\vartheta')||[DF_0 \circ \mathcal{J}]^{(r,\vartheta)}_{(r',\vartheta')}|.
\end{aligned}$$

By (4.19) and Lemma 4.8, we obtain

$$\begin{aligned}
&|[D\mathcal{J}]^{(r,\vartheta)}_{(r',\vartheta')}||DF_0(\mathcal{J}(r,\vartheta))| \le C\, d^{-\alpha}[D\mathcal{J}]^{(0,\Gamma_w)}_{\alpha,\mathcal{N}_{r_s}^+}\, d^{-1+\alpha}K_{F_0},\\
&|D\mathcal{J}(r',\vartheta')||[DF_0 \circ \mathcal{J}]^{(r,\vartheta)}_{(r',\vartheta')}| \le C\, d^{-1}K_{\mathcal{J}}\, K_{F_0}(K_{\mathcal{J}}|(r,\vartheta)-(r',\vartheta')|)^\alpha
\end{aligned}$$

with $K_{\mathcal{J}} = |\mathcal{J}|_{1,\mathcal{N}_{r_s}^+}$ and $K_{F_0} = \|F_0\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda}$. Thus $F = F_0 \circ \mathcal{J}$ satisfies (4.13).

The uniqueness of a solution can be checked easily. If $F^{(1)}$ and $F^{(2)}$ are solutions to (4.12) then $F^{(1)} - F^{(2)}$ is identically 0 along every characteristic $X(t;r,\vartheta)$ associated with W, and so $F^{(1)} = F^{(2)}$ holds in $\mathcal{N}_{r_s}^+$. □

5. Proof of Theorem 1:Existence

To prove Theorem 1, we prove a weak implicit mapping theorem for infinite dimensional Banach spaces.

5.1. **Weak implicit mapping theorem.** Fix a constant $R > 0$ and set

$$\mathcal{B}^{(2)}_R(h_0) := \{h \in C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda) : \|h - h_0\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda} \le R\}. \tag{5.1}$$

For $\mathcal{B}^{(1)}_\sigma(Id, \varphi_0^-, p_0^-, v_c)$ in (3.30), define $\mathcal{P} : \mathcal{B}^{(1)}_\sigma(Id, \varphi_0^-, p_0^-, v_c) \to \mathcal{B}^{(2)}_{C\sigma}(p_c)$ by

$$\mathcal{P} : (\Psi, \varphi_-, p_-, v_{ex}) \mapsto p|_{\Gamma_{ex}} \tag{5.2}$$

where p is the solution to (2.43) with φ in Proposition 3.1 for the given $(\Psi, \varphi_-, p_-, v_{ex})$. By Proposition 3.1 and 4.1, there exists positive constants σ_3 and C so that whenever $0 < \sigma \le \sigma_3$,

$\mathcal{P}$, defined in (5.2), is well defined. To prove theorem 1, we need to show $\mathcal{P}^{-1}(p_{ex}) \neq \emptyset$ for any $p_{ex} \in \mathcal{B}^{(2)}_{\tilde{\sigma}}(p_c)$ for a sufficiently small constant $\tilde{\sigma} > 0$. So, we need the following lemma.

Proposition 5.1 (Weak Implicit Mapping Theorem). *Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be Banach spaces compactly imbedded in Banach spaces $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively. Also, suppose that there are two Banach spaces $\mathfrak{C}_3, \mathfrak{B}_3$ with $\mathfrak{C}_3 \subset \mathfrak{B}_3$. For a point $(x_0, y_0) \in \mathfrak{C}_1 \times \mathfrak{C}_2$, suppose that a mapping $\mathcal{F}$ satisfies the followings:*

(i) *$\mathcal{F}$ maps a small neighborhood of (x_0, y_0) in $\mathfrak{B}_1 \times \mathfrak{B}_2$ to $\mathfrak{B}_3$, and maps a neighborhood of (x_0, y_0) in $\mathfrak{C}_1 \times \mathfrak{C}_2$ to $\mathfrak{C}_3$ with $\mathcal{F}(x_0, y_0) = \mathbf{0}$,*
(ii) *whenever a sequence $\{(x_k, y_k)\} \subset \mathfrak{C}_1 \times \mathfrak{C}_2$ near (x_0, y_0) converges to (x_*, y_*) in $\mathfrak{B}_1 \times \mathfrak{B}_2$, the sequence $\{\mathcal{F}(x_k, y_k)\} \subset \mathfrak{C}_3$ converges to $\mathcal{F}(x_*, y_*)$ in $\mathfrak{B}_3$,*
(iii) *$\mathcal{F}$, as a mapping from $\mathfrak{B}_1 \times \mathfrak{B}_2$ to $\mathfrak{B}_3$ also as a mapping from $\mathfrak{C}_1 \times \mathfrak{C}_2$ to $\mathfrak{C}_3$, is Fréchet differentiable at (x_0, y_0),*
(iv) *the partial Fréchet derivative $D_x\mathcal{F}(x_0, y_0)$, as a mapping from $\mathfrak{B}_1 \times \mathfrak{B}_2$ to $\mathfrak{B}_3$ also as a mapping from $\mathfrak{C}_1 \times \mathfrak{C}_2$ to $\mathfrak{C}_3$, is invertible.*

Then there is a small neighborhood $\mathcal{U}_2(y_0)$ of y_0 in $\mathfrak{C}_2$ so that, for any given $y \in \mathcal{U}_2(y_0)$, there exists $x_* = x_*(y)$ satisfying $\mathcal{F}(x^*(y), y) = \mathbf{0}$.

In [SZ], the author used the Brouwer fixed point theorem to prove *the right inverse function theorem* for finite dimensional Banach spaces without assuming continuous differentiability of a mapping. As an analogy, we apply the Schauder fixed point theorem to prove Proposition 5.1. The detailed proof is given in Appendix A.

To apply Proposition 5.1 to $\mathcal{P}$, we need to verify that $\mathcal{P}$ is continuous, Fréchet differentiable at $(Id, \varphi_0^-, p_0^-, v_c)$ and a partial Fréchet derivative of $\mathcal{P}$ at the point is invertible.

5.2. **Fréchet differentiability of $\mathcal{P}$ at $\zeta_0 = (Id, \varphi_0^-, p_0^-, v_c)$.** To prove Fréchet differentiability of $\mathcal{P}$ at ζ_0, it suffices to consider how to compute the partial Fréchet derivative of $\mathcal{P}$ with respect to v_{ex} at ζ_0 because the other partial derivatives of $\mathcal{P}$ can be obtained similarly.

For $\mathcal{P}$ defined in (5.2), let us define $\mathcal{R}$ and $\mathcal{Q}$ by

$$\begin{aligned} &\mathcal{R} : (\Psi, \varphi_-, p_-, v_{ex}) \mapsto (B_0 - \frac{1}{2}|D\Psi^{-1}D\varphi|^2)^{\frac{\gamma}{\gamma-1}}|_{\Gamma_{ex}} \\ &\mathcal{Q} : (\Psi, \varphi_-, p_-, v_{ex}) \mapsto \frac{\mathcal{P}(\Psi, \varphi_-, p_-, v_{ex})}{\mathcal{R}(\Psi, \varphi_-, p_-, v_{ex})}. \end{aligned} \tag{5.3}$$

If $\mathcal{R}$ and $\mathcal{Q}$ are Fréchet differentiable at ζ_0 then the Fréchet derivative $D\mathcal{P}$ of $\mathcal{P}$ at ζ_0 is given by

$$D\mathcal{P}_{\zeta_0} = \mathcal{R}(\zeta_0)D\mathcal{Q}_{\zeta_0} + \mathcal{Q}(\zeta_0)D\mathcal{R}_{\zeta_0}. \tag{5.4}$$

To simplify notations, we denote as $D_v\mathcal{P}, D_v\mathcal{Q}$ and $D_v\mathcal{R}$ respectively for the partial Fréchet derivatives of $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ at ζ_0 with respect to v_{ex} hereafter.

Lemma 5.2. *Fix any $\alpha \in (0, 1)$, and set $\mathfrak{A} := C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)$. As mappings from $\mathcal{B}^{(1)}_{\sigma}(\zeta_0)$ to $\mathcal{B}^{(2)}_{C\sigma}(p_c)$, $\mathcal{Q}, \mathcal{R}$ and $\mathcal{P}$ are Fréchet differentiable at ζ_0. In particular, the partial Fréchet derivatives of $\mathcal{Q}, \mathcal{R}$ and $\mathcal{P}$ at ζ_0 with respect to v_{ex} are given by*

$$D_v\mathcal{R} : w \in \mathfrak{A} \mapsto a_1 w \in \mathfrak{A} \tag{5.5}$$

$$D_v\mathcal{Q}w \in \mathfrak{A} \mapsto a_2\psi_0(r_s, \cdot) \in \mathfrak{A} \tag{5.6}$$

$$D_v\mathcal{P}w \in \mathfrak{A} \mapsto \mathcal{Q}(\zeta_0)a_1 w + \mathcal{R}(\zeta_0)a_2\psi_0(r_s, \cdot) \in \mathfrak{A} \tag{5.7}$$

for any $w \in \mathfrak{A}$ *with*

$$\begin{aligned} a_1 &= -\frac{2\gamma(B_0 - \frac{1}{2}|\partial_r\varphi_0^+|^2)^{\frac{\gamma}{\gamma-1}}\partial_r\varphi_0^+}{(\gamma+1)r_1^{n-1}(K_0 - |\partial_r\varphi_0^+|^2)}|_{r=r_1} \\ a_2 &= \frac{1}{(B_0 - \frac{1}{2}|\nabla\varphi_0^+(r_s)|^2)^{\frac{\gamma}{\gamma-1}}}\Big(\frac{\frac{d}{dr}(p_{s,0}-p_0^+)}{\partial_r(\varphi_0^- - \varphi_0^+)} + \frac{\gamma p_0^+\mu_0\partial_r\varphi_0^+}{(\gamma-1)(B_0 - \frac{1}{2}|\partial_r\varphi_0^+|^2)}|_{r=r_s}\Big) \end{aligned} \tag{5.8}$$

with μ_0 *defined in Lemma 2.5, and*

$$p_{s,0} = \rho_0^-|\nabla\varphi_0^- \cdot \hat{r}|^2 + p_0^- - \rho_0^- K_0 \tag{5.9}$$

where K_0 *is defined in* (2.30). *Here,* ψ_0 *is a unique solution to an elliptic boundary problem specified in the proof.*

Proof. In this proof, the constants ε_0 and C appeared in various estimates depend on the data in sense of Remark 2.7.

Since (5.7) follows from (5.5) and (5.6), it suffices to prove (5.5) and (5.6).

If a bounded linear mapping $L : \mathfrak{A} \to \mathfrak{A}$ satisfies

$$\|\mathcal{Q}(Id, \varphi_0^-, p_0^-, v_c + \varepsilon w) - \mathcal{Q}(Id, \varphi_0^-, p_0^-, v_c) - \varepsilon Lw\|_{1,\alpha,\Lambda}^{(-\alpha,\partial\Lambda)} \le o(\varepsilon) \tag{5.10}$$

for any $w \in \mathfrak{A}$ with $\|w\|_{1,\alpha,\Lambda}^{(-\alpha,\partial\Lambda)} = 1$ and if $o(\varepsilon)$ is independent of w, then we have $L = D_v\mathcal{Q}$. To find such L, we fix $w \in \mathfrak{A}$ with $\|w\|_{1,\alpha,\Lambda}^{(-\alpha,\partial\Lambda)} = 1$, and compute the Gâteaux derivative $\frac{d}{d\varepsilon}|_{\varepsilon=0}\mathcal{Q}(Id, \varphi_0^-, p_0^-, v_c + \varepsilon w)$.

Fix a sufficiently small constant ε_0, and, for $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$, let φ_ε be a solution indicated in Proposition 3.1 with $\Psi = Id, (\varphi_-, p_-) = (\varphi_0^-, p_0^-)$ and $v_{ex} = v_c + \varepsilon w$ on Γ_{ex} in (3.1).

Let us denote the subsonic domain of φ_ε as $\mathcal{N}_\varepsilon^+$, the transonic shock as $S_\varepsilon = \{r = f_\varepsilon(\vartheta)\}$, Φ_{f_ε} as Φ_ε, and G_{φ_ε} as G_ε where Φ_{f_ε} and G_{φ_ε} are defined in (3.44) and (4.3). For W_ε^*, defined as in (4.9), let $E_\varepsilon \in C^{1,\alpha}_{(-\alpha,\Gamma_w)}(\mathcal{N}_{r_s}^+)$ be a solution to

$$W_\varepsilon^* \cdot DE_\varepsilon = 0 \text{ in } \mathcal{N}_{r_s}^+ \tag{5.11}$$

$$E_\varepsilon(r_s, \vartheta) = \frac{p_{s,\varepsilon}}{(B_0 - \frac{1}{2}|\nabla\varphi_\varepsilon|^2)^{\frac{\gamma}{\gamma-1}}}(f_\varepsilon(\vartheta), \vartheta) \tag{5.12}$$

with $p_{s,\varepsilon} = \rho_0^-(D\varphi_0^- \cdot \nu_{s,\varepsilon})^2 + p_0^- - \rho_0^- K_{s,\varepsilon}$ where $D = (\partial_r, \partial_{\vartheta_1}, \cdots, \partial_{\vartheta_{n-1}})$ and $\nu_{s,\varepsilon}$ is the unit normal vector field on S_ε toward $\mathcal{N}_\varepsilon^+$, and $K_{s,\varepsilon}$ is defined by (2.51) with $(\rho_-, \varphi_-, p_-) = (\rho_0^-, \varphi_0^-, p_0^-)$ and $\Psi = Id$. By Proposition 4.1, the solution E_ε to (5.11) with (5.12) is unique, and (5.3) implies $\mathcal{Q}(Id, \varphi_0^-, p_0^-, v_c + \varepsilon w) = E_\varepsilon|_{\Gamma_{ex}}$. Similarly, $\mathcal{Q}(\zeta_0)$ is the exit value of the solution E_0^+ to

$$W_0^* \cdot DE_0^+ = 0 \;\text{ in }\; \mathcal{N}_{r_s}^+, \quad E_0^+(r_s, \vartheta) = \frac{p_{s,0}(r_s)}{(B_0 - \frac{1}{2}|\partial_r\varphi_0^+(r_s)|^2)^{\frac{\gamma}{\gamma-1}}}$$

where W_0^* is defined by (4.9) with $(\Psi, \varphi, \rho_-, \varphi_-, p_-) = (Id, \varphi_0^+, \rho_0^-, \varphi_0^-, p_0^-)$, and $p_{s,0}$ is defined in (5.9). Since φ_0^+ is a radial function, we have $W_0^* = (1, 0, \cdots, 0) = \hat{r}$, so E_0^+ is a constant in $\mathcal{N}_{r_s}^+$, and this implies

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}\mathcal{Q}(Id, \varphi_0^-, p_0^-, v_c + \varepsilon w) = \frac{d}{d\varepsilon}|_{\varepsilon=0}E_\varepsilon(r_1, \cdot) = \frac{d}{d\varepsilon}|_{\varepsilon=0}(E_\varepsilon - E_0^+)(r_1, \cdot). \tag{5.13}$$

Note that, by Proposition 3.1 and Lemma 4.2, if $\varepsilon_0 > 0$ is sufficiently small, then whenever $0 < |\varepsilon| < \varepsilon_0$, there exists a unique $\psi_\varepsilon \in C^{2,\alpha}(\mathcal{N}_\varepsilon^+)$ to satisfy

$$\varphi_\varepsilon = \varphi_0^+ + \varepsilon\psi_\varepsilon \text{ in } \mathcal{N}_\varepsilon^+,$$

and moreover, there is a constant $C > 0$ so that there holds

$$\begin{aligned}
&\|\psi_\varepsilon\|_{2,\alpha,\mathcal{N}_\varepsilon^+}^{(-1-\alpha,\overline{S}_\varepsilon\cup\overline{\Gamma}_{ex})} \le C,\\
&\|W_\varepsilon^* - W_0^*\|_{1,\alpha,\mathcal{N}_{r_s}^+}^{(-1-\alpha,\Gamma_w)} + \|W_\varepsilon^* - W_0^*\|_{1,\alpha,\mathcal{N}_{r_s}^+}^{(-1-\alpha,\overline{S}_0\cup\overline{\Gamma}_{ex})} \le C|\varepsilon|.
\end{aligned} \tag{5.14}$$

By (5.13), we formally differentiate $W_\varepsilon \cdot D(E_\varepsilon - E_0^+) = 0$ with respect to ε, and apply (4.1),(4.2) and (5.14) to obtain

$$W_0 \cdot \frac{d}{d\varepsilon}|_{\varepsilon=0}(E_\varepsilon - E_0^+) = 0 \quad \text{in } \mathcal{N}_{r_s}^+. \tag{5.15}$$

Then (5.15) implies

$$\begin{aligned}
&\frac{d}{d\varepsilon}|_{\varepsilon=0}\mathcal{Q}(Id,\varphi_0^-,p_0^-,v_c+\varepsilon w) = \frac{d}{d\varepsilon}|_{\varepsilon=0}(E_\varepsilon - E_0^+)(r_s)\\
&= \frac{d}{d\varepsilon}|_{\varepsilon=0}\Big(\frac{p_{s,\varepsilon}}{(B_0-\frac{1}{2}|D\varphi_\varepsilon|^2)^{\frac{\gamma}{\gamma-1}}} - \frac{p_0^+}{(B_0-\frac{1}{2}|D\varphi_0^+|^2)^{\frac{\gamma}{\gamma-1}}}\Big)|_{(f_\varepsilon(\theta),\theta)}
\end{aligned} \tag{5.16}$$

where $p_{s,\varepsilon}$ is given in (5.12).

Set $p_s(\varepsilon, f_\varepsilon(\theta),\theta) := p_{s,\varepsilon}(f_\varepsilon(\theta),\theta)$. By (5.14), (2.49) and (2.52), $p_s(\varepsilon,\cdot,\cdot)$ is partially differentiable with respect to ε at $\varepsilon = 0$. Then this provides

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}(p_{s,\varepsilon}-p_0^+)(f_\varepsilon(\theta),\theta) = \frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}p_s(\varepsilon,r_s,\theta)+(p_{s,0}-p_0^+)'(r_s)\frac{d}{d\varepsilon}|_{\varepsilon=0}f_\varepsilon(\theta). \tag{5.17}$$

First of all, An explicit calculation using (2.52) and (5.21) implies

$$\frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}p_s(\varepsilon,r_s,\theta) = 0. \tag{5.18}$$

Set $\tilde\psi_\varepsilon := \psi_\varepsilon \circ \Phi_\varepsilon^{-1}$ then (3.44) implies $\psi_\varepsilon(f_\varepsilon(\vartheta),\vartheta) = \tilde\psi_\varepsilon(r_s,\vartheta)$. We note that $\varepsilon\tilde\psi_\varepsilon$ solves (3.45) with $\phi = \varphi_0^+ + \varepsilon\psi_\varepsilon(=\varphi_\varepsilon)$ and $\tilde g_3 = \varepsilon w$. Since all the coefficients, including $\widetilde{F}, \tilde g_1$ and $\tilde g_2$ in (3.45) smoothly depend on ϕ and $D\phi$, $\tilde\psi_\varepsilon$ converges to ψ_0 in the norm of $C^{2,\alpha}_{(-1-\alpha,\Gamma_w)}(\mathcal{N}_{r_s}^+)$ where ψ_0 is a unique solution to

$$\partial_j(a_{jk}(x,0)\partial_k\psi_0) = 0 \text{ in } \mathcal{N}_{r_s}^+, \qquad D\psi_0\cdot\hat r - \mu_0\psi_0 = 0 \text{ on } S_0 \tag{5.19}$$

$$(a_{jk}(x,0)\partial_k\psi_0)\cdot\nu_w = 0 \text{ on } \Gamma_{w,r_s}, \quad (a_{jk}(x,0)\partial_k\psi_0)\cdot\hat r = w \text{ on } \Gamma_{ex}, \tag{5.20}$$

where a_{jk}, μ_0 are defined in (3.6) and Lemma 2.5, Also, by a method similar to the proof of Lemma 3.11, we have

$$\|\psi_\varepsilon\circ\Phi_\varepsilon^{-1} - \psi_0\|_{2,\alpha,\mathcal{N}_{r_s}^+}^{(-1-\alpha,\Gamma_w)} \le C|\varepsilon|. \tag{5.21}$$

By (2.47), for any $\varepsilon \in [-\varepsilon_0,\varepsilon_0]\setminus\{0\}$, we have $(\varphi_0^- - \varphi_0^+)(f_\varepsilon(\vartheta)) = \varepsilon\psi_\varepsilon(f_\varepsilon(\vartheta),\vartheta)$. Combining this with (5.17), (5.21) and (5.18), we obtain

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}(p_{s,\varepsilon}-p_0^+)(f_\varepsilon(\theta),\theta) = \frac{(p_{s,0}-p_0^+)(r_s)}{\partial_r(\varphi_0^- - \varphi_0^+)(r_s)}\psi_0(r_s,\theta). \tag{5.22}$$

Similarly, by using (5.19), (5.21), we also obtain

$$\begin{aligned}
&\frac{d}{d\varepsilon}|_{\varepsilon=0}\Big(\frac{1}{(B_0-\frac{1}{2}|\nabla\varphi_\varepsilon|^2)^{\frac{\gamma}{\gamma-1}}}-\frac{1}{(B_0-\frac{1}{2}|\partial_r\varphi_0^+|^2)^{\frac{\gamma}{\gamma-1}}}\Big)(f_\varepsilon(\vartheta),\vartheta)\\
&=\frac{\gamma\partial_r\varphi_0^+(r_s)}{(\gamma-1)(B_0-\frac{1}{2}(\partial_r\varphi_0^+(r_s))^2)^{\frac{2\gamma-1}{\gamma-1}}}\partial_r\psi_0(r_s,\vartheta)\\
&=\frac{\gamma\partial_r\varphi_0^+(r_s)\mu_0}{(\gamma-1)(B_0-\frac{1}{2}(\partial_r\varphi_0^+(r_s))^2)^{\frac{2\gamma-1}{\gamma-1}}}\psi_0(r_s,\vartheta)=:a_3\psi_0(r_s,\vartheta).
\end{aligned}\tag{5.23}$$

Set $Lw:=a_2\psi_0(r_s,\cdot)$ for a_2 defined in (5.8).

Since the constant C in (4.1),(4.2), (5.14) and (5.21) only depends on the data in sense of Remark 2.7 , combining (5.22) with (5.23), we conclude that the mapping L satisfies (5.10) with $o(\varepsilon)$ being independent of w. Thus we conclude $D_v\mathcal{Q}=L$. Since it is easier to verify (5.5) so we omit the proof. □

Corollary 5.3. $D_v\mathcal{Q}:C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)\to C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)$ *is a compact mapping.*

Proof. For any $w\in C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)$, let ψ_0 be a unique solution to (5.19), (5.20). Then there are constants $\widetilde{C},C>0$ depending only on the data to satisfy

$$\|D_v\mathcal{Q}w\|^{(-1-\alpha,\partial\Lambda)}_{2,\alpha,\Lambda}\le\widetilde{C}\|\psi_0\|^{(-1-\alpha,\Gamma_w)}_{2,\alpha,\mathcal{N}^+_{r_s}}\le C\|w\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda}.$$

□

5.3. **Local invertibility of $\mathcal{P}$ near $\zeta_0=(Id,\varphi_0^-,p_0^-,v_c)$.** For a local invertibility of $\mathcal{P}$, by Lemma 5.1, it remains to show that $D_v\mathcal{P}$ is invertible and $\mathcal{P}$ is continuous in a Banach space.

Note that for any $\alpha\in(0,1)$, $\mathcal{B}^{(1)}_\sigma(\zeta_0)$ and $\mathcal{B}^{(2)}_{C\sigma}(p_c)$ are compact and convex subsets of $\mathfrak{B}_{(1)}:=C^{2,\alpha/2}(\mathcal{N},\mathbb{R}^n)\times C^{3,\alpha/2}(\mathcal{N}^-_{r_s+\delta})\times C^{2,\alpha/2}(\mathcal{N}^-_{r_s+\delta})\times C^{1,\alpha/2}_{-\alpha/2,\partial\Lambda}(\Lambda)$ and $\mathfrak{B}_{(2)}:=C^{1,\alpha/2}_{(-\alpha/2,\partial\Lambda)}(\Lambda)$ respectively. We define, for $z=(z_1,z_2,z_3,z_4)$,

$$\begin{aligned}
&\|z\|_{(1)}:=\|z_1\|_{2,\alpha/2,\mathcal{N}}+\|z_2\|_{3,\alpha/2,\mathcal{N}^-_{r_s+\delta}}+\|z_3\|_{2,\alpha/2,\mathcal{N}^-_{r_s+\delta}}+\|z_4\|^{(-\alpha/2,\partial\Lambda)}_{1,\alpha/2,\Lambda}\\
&\|z_4\|_{(2)}:=\|z_4\|^{(-\alpha/2,\partial\Lambda)}_{1,\alpha/2,\Lambda},
\end{aligned}\tag{5.24}$$

and regard $\mathcal{P}$ as a mapping from a subset of $\mathfrak{B}_{(1)}$ to a subset of $\mathfrak{B}_{(2)}$.

Lemma 5.4. *For any $\alpha\in(0,1)$, $\mathcal{P}:\mathcal{B}^{(1)}_\sigma(\zeta_0)\to\mathcal{B}_{C\sigma}(p_c)$ is continuous in sense that if ζ_j converges to ζ_∞ in $\mathfrak{B}_{(1)}$ as j tends to ∞, then $\mathcal{P}(\zeta_j)$ converges to $\mathcal{P}(\zeta_\infty)$ in $\mathfrak{B}_{(2)}$.*

Proof. Take a sequence $\{\zeta_j=(\Psi_j,\varphi_-^{(j)},p_-^{(j)},v_j)\}$ in $\mathcal{B}^{(1)}_\sigma(\zeta_0)$ so that ζ_j converges to $\zeta_\infty\in\mathcal{B}^{(1)}_\sigma(\zeta_0)$ in $\mathfrak{B}_{(1)}$ as j tends to ∞. For each $j\in\mathbb{N}$, let E_j and E_∞ be the solutions to the transport equation of (4.8) with $W^*=W^*_j$ and $W^*=W^*_\infty$ respectively in (4.9) where the initial conditions $E_j(r_s,\vartheta),E_\infty(r_s,\vartheta)$ in (4.10) are determined by the data ζ_j and ζ_∞ respectively. Then, we have $\mathcal{Q}(\zeta_j)=E_j|_{\Gamma_{ex}}$ and $\mathcal{Q}(\zeta_\infty)=E_\infty|_{\Gamma_{ex}}$. By the convergence of ζ_j in $\mathfrak{B}_{(1)}$, W^*_j converges to W^*_∞ in $C^{1,\alpha/2}_{(-\alpha/2,\overline{S_0}\cup\overline{\Gamma_{ex}})}(\mathcal{N}^+_{r_s})\cap C^{1,\alpha/2}_{(-\alpha/2,\Gamma_w)}(\mathcal{N}^+_{r_s})$ as j tends to ∞. Because of $\{\zeta_j\}\subset\mathcal{B}^{(1)}_\sigma(\zeta_0)$, by Proposition 3.1 and 4.1, $\{E_j\}$ is bounded in $C^{1,\alpha}_{(-\alpha,\Gamma_w)}(\mathcal{N}^+_{r_s})$ so there is a

subsequence of $\{E_j\}$, which we simply denote as $\{E_j\}$ itself, that converges to, say $\widetilde{E}_\infty$, in $C^{1,\alpha/2}_{(-\alpha/2,\Gamma_w)}(\mathcal{N}^+_{r_s})$. Then $\widetilde{E}_\infty$ satisfies

$$W^*_\infty \cdot D\widetilde{E}_\infty = 0 \text{ in } \mathcal{N}^+_{r_s}, \quad \widetilde{E}_\infty(r_s,\vartheta) = E_\infty(r_s,\vartheta) \text{ for all } \vartheta\in\Lambda,$$

and thus $\widetilde{E}_\infty = E_\infty$ holds in $\mathcal{N}^+_{r_s}$. Moreover, this holds true for any subsequence of $\{E_j\}$ that converges in $C^{1,\alpha/2}_{(-\alpha/2,\Gamma_w)}(\mathcal{N}^+_{r_s})$. Therefore, we conclude that $\mathcal{Q}(\zeta_j)$ converges to $\mathcal{Q}(\zeta_\infty)$ in $\mathfrak{B}_{(2)}$ i.e., $\mathcal{Q}$ is continuous. One can similarly prove that $\mathcal{R}$ is continuous. By the definition of $\mathcal{P}$ in (5.2), we finally conclude that $\mathcal{P}$ is continuous. □

To prove the invertibility of $D_v\mathcal{P} : C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda) \to C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)$, let us make the following observation.

For a given $w\in C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)$, let $\psi^{(w)}$ be the solution to (5.19),(5.20). Define a mapping T by $T : w\mapsto -\frac{a_2}{a_1}\psi^{(w)}(r_s,\cdot)$, then T maps $C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)$ into itself, and we can write $D_v\mathcal{P}$ in Lemma 5.2 as

$$D_v\mathcal{P}w = a_1(I-T)w. \tag{5.25}$$

By Corollary 5.3, T is compact. Then, by the Fredholm alternative theorem, either $D_v\mathcal{P}w = 0$ has a nontrivial solution w or $D_v\mathcal{P}$ is invertible. In the following lemma,we show that $D_v\mathcal{P}$ satisfies the latter hence $D_v\mathcal{P}$ is invertible.

Lemma 5.5. *For $\alpha\in(0,1)$, $D_v\mathcal{P} : C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda) \to C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)$ is invertible.*

Proof. The proof is divided into two steps.

Step 1. Let a_1 and a_2 be defined as in (5.8). We claim $a_2<0$ and $-\frac{a_2}{a_1}<0$.

By Lemma 2.5, we can write a_2 as

$$a_2 = \frac{\partial_r(p_{s,0}-p_0^+) - \rho_0^-\partial_r\varphi_0^-\partial_r(\frac{K_0}{\partial_r\varphi_0^-} - \partial_r\varphi_0^+)}{\partial_r(\varphi_0^- - \varphi_0^+)(r_s)}\,|_{r=r_s}\,.$$

Then, by (2.34), (2.36) and (3.19), we obtain

$$a_2 = \frac{(n-1)\rho_0^-(r_s)(K_0 - (\partial_r\varphi_0^-(r_s))^2)}{r_s(B_0 - \frac{1}{2}|D\varphi_0^+(r_s)|)^{\frac{\gamma}{\gamma-1}}\partial_r(\varphi_0^- - \varphi_0^+)(r_s)} < 0.$$

Since $a_1<0$ is obvious from (5.7), there holds $-\frac{a_2}{a_1}<0$.

Step 2. *Claim. $D_v\mathcal{P}w=0$ if and only if $w=0$.*

If $w=0$ then it is obviously $D_v\mathcal{P}w$=0.

According to Remark 3.10, in spherical coordinates, (5.19) and (5.20) are expressed as

$$\begin{aligned}
&\partial_r\{k_1(r)\partial_r\psi\} + k_2(r)\triangle_{S^{n-1}}\psi = 0 \text{ in } \mathcal{N}^+_{r_s}, && \partial_r\psi - \mu_0\psi = 0 \text{ on } S_0,\\
&\partial_{\nu_w}\psi = 0 \text{ on } \Gamma_{w,r_s}, && \partial_r\psi = \frac{w}{k_1(r_1)} \text{ on } \Gamma_{ex}
\end{aligned} \tag{5.26}$$

where k_1 and k_2 are defined as in (3.52).

Suppose $D_v\mathcal{P}w=0$ for some $w\in C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)$. Then, by (5.25), there holds

$$w(\vartheta) = -\frac{a_2}{a_1}\psi^{(w)}(r_s,\vartheta) \text{ for all } \vartheta\in\Lambda. \tag{5.27}$$

We denote $\psi^{(w)}$ as ψ for the simplicity hereafter. Let ν_* be a unit normal vector field on $\partial\Lambda\subset\mathbb{S}^{n-1}$. Then ν_* is perpendicular to Γ_w so w satisfies

$$\partial_{\nu_*}w=-\frac{a_2}{a_1}\partial_{\nu_w}\psi(r_s,\cdot)=0\ \text{ on }\partial\Lambda.$$

Then w has an eigenfunction expansion

$$w=\sum_{j=1}^{\infty}c_j\eta_j\ \text{ in }\ \Lambda\ \text{ with }\ c_j=\int_{\Lambda}w\eta_j$$

where every η_j is an eigenfunction of the eigenvalue problem

$$-\triangle_{\mathbb{S}^{n-1}}\eta_j=\lambda_j\eta_j\ \text{ in }\ \Lambda,\qquad \partial_{\nu_*}\eta_j=0\ \text{ on }\ \partial\Lambda \tag{5.28}$$

with $\|\eta_j\|_{L^2(\Lambda)}=1$. We note that all the eigenvalues of (5.28) are non-negative real numbers.

We claim that (5.27) implies $c_j=0$ for all $j\in\mathbb{N}$. Once this is verified, then we obtain $w=0$.

For each $N\in\mathbb{N}$, let us set $w_N:=\sum_{j=1}^N c_j\eta_j$ and $\psi_N(r,\vartheta):=\sum_{j=1}^N q_j(r)\eta_j(\vartheta)$ for one variable functions q_j satisfying

$$(k_1q_j')'-\lambda_jk_2q_j=0\ \text{ in }\ (r_s,r_1), \tag{5.29}$$

$$q_j'(r_s)-\mu_0q_j(r_s)=0,\quad q_j'(r_1)=\frac{c_j}{k_1(r_1)}. \tag{5.30}$$

for $j=1,\cdots,N$. Then $\psi_N(r_s,\cdot)$ converges to $\psi(r_s,\cdot)$ in $L^2(\Lambda)$ so that (5.27) implies

$$q_j(r_s)=-\frac{a_1}{a_2}c_j\ \text{ for all }\ j\in\mathbb{N}.$$

Suppose that $c_j\neq 0$ for some $j\in\mathbb{N}$. Without loss of generality, we assume $c_j>0$. By the sign of $\frac{a_1}{a_2}$ considered in Step 1, we have $q_j'(r_s)<0, q_j(r_s)<0$ and $q_j'(r_1)>0$. We will derive a contradiction.

(i) If $\lambda_j=0$ then there is a constant m_0 so that $k_1q_j'=m_0$ for all $r\in[r_s,r_1]$. Since k_1 is strictly positive in $[r_s,r_1]$, we have $sgn\ m_0=sgn\ q_k'(r_1)$ and $sgn\ m_0=sgn\ q_j'(r_s)$. Then $sgn\ q_j'(r_s)=sgn\ q_j'(r_1)$ must hold. But this contradicts to the assumption of $q_j'(r_s)<0<q_j'(r_1)$.

(ii) Assume $\lambda_j>0$. By (5.29), we have

$$\int_{r_s}^{r_1}\lambda_jk_2q_jdr=\int_{r_s}^{r_1}(k_1q_j')'dr=k_1(r_1)q_j'(r_1)-k_1(r_s)q_j'(r_s)>0.$$

Then $q_j(r_s)<0$ implies $\max_{[r_s,r_1]}q_j>0$ so there exists $t_1\in[r_s,r_1]$ satisfying $q_j(t_1)=0$. Also, by the intermediate value theorem, there exists $t_2\in[r_s,r_1]$ satisfying $q_j'(t_2)=0$.

If $t_1=t_2$ then the uniqueness theorem of second-order linear ODEs implies $q_j\equiv 0$ on $[r_s,r_1]$. But this is a contradiction. If $t_1\neq t_2$, assume $t_1<t_2$ without loss of generality. Then, by the maximum principle, $q_j\equiv 0$ on $[t_1,t_2]$ so $q_j\equiv 0$ on $[r_s,r_1]$. This is a contradiction as well.

Thus every c_j must be 0 hence $w=0$. By the Fredholm alternative theorem, we finally conclude that $D_v\mathcal{P}$ is invertible. $\square$

Proposition 5.6. *For any $\alpha \in (0,1)$, there exists a positive constant $\sigma_1(\leq \sigma_3)$ depending on the data in sense of Remark 2.7 so that whenever $0 < \sigma \leq \sigma_1$, for a given $(\Psi, \varphi_-, p_-, p_{ex})$ with $\varsigma_1, \varsigma_2, \varsigma_3$ in Theorem 1 satisfying $0 \leq \varsigma_l \leq \sigma$ for $l = 1,2,3$, there exists a function $v_{ex}^* = v_{ex}^*(\Psi, \varphi_-, p_-, p_{ex}) \in C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)$ so that there holds*

$$\mathcal{P}(\Psi, \varphi_-, p_-, v_{ex}^*) = p_{ex}. \tag{5.31}$$

Moreover such a v_{ex}^* satisfies

$$\|v_{ex}^* - v_c\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda} \leq C(\varsigma_1 + \varsigma_2 + \varsigma_3). \tag{5.32}$$

Proof. Following the definition (3.30), define, for the constant C in (4.1), $\mathcal{P}_* : \mathcal{B}^* := \mathcal{B}^{(1)}(\zeta_0) \times \mathcal{B}^{(13)}_{C\sigma}(p_c) \to \mathcal{B}^{(13)}_{2C\sigma}(0)$ by

$$\mathcal{P}_*(\Psi, \varphi_-, p_-, v_{ex}, p_{ex}) := \mathcal{P}(\Psi, \varphi_-, p_-, v_{ex}) - p_{ex}.$$

Then, by Lemma 5.4 and the definition of $\mathcal{P}_*$ above, for $\{(\zeta-_j, p_j)\} \subset \mathcal{B}^*$, if $\lim_{j\to\infty} \|\zeta_j - \zeta_\infty\|_{(1)} + \|p_j - p_\infty\|_{(2)} = 0$ for some $(\zeta_\infty, p_{infty}) \in \mathcal{B}^*$ where $\|\cdot\|_{k=1,2}$ are defined in (5.24), then $\lim_{j\to\infty} \|\mathcal{P}_*(\zeta_j, p_j) - \mathcal{P}_*(\zeta_\infty, p_\infty)\| = 0$ holds, i.e., $\mathcal{P}_*$ is continuous in the sense similar to Lemma 5.4. Besides, by Lemma 5.2, $\mathcal{P}_*$ is Fréchet differentiable at (ζ_0, p_c). In particular, we have $D_v\mathcal{P}_*(\zeta_0, p_c) = D_v\mathcal{P}$ so it is invertible. So if $\sigma > 0$ is sufficiently small depending on the data and α then Lemma 5.1 implies (5.31). (5.32) is a direct result from Remark A.2. □

Proof of Theorem 1. Fix $\alpha \in (0,1)$. By Proposition 5.6, for a given $\Psi, \varphi_-, p_-, p_{ex}$ with $\varsigma_1, \varsigma_2, \varsigma_3$ in Theorem 1 satisfying $0 \leq \varsigma_l \leq \sigma_1$ for $l = 1,2,3$, there exists a $v_{ex}^* \in C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)$ so that there holds (5.31). For such a v_{ex}^*, Proposition 3.1 and 4.1 imply that there exists a unique (ρ^*, φ^*, p^*) in $\mathcal{N}^+$ satisfying (2.44), (3.1), Proposition 3.1, Lemma 4.2 and Proposition 4.1. So $(\rho, \varphi, p) = (\rho^*, \varphi^*, p^*)\chi_{\mathcal{N}^+} + (\rho_-, \varphi_-, p_-)\chi_{\mathcal{N}\setminus\mathcal{N}^+}$ is a transonic shock solution satisfying Theorem (1) (a)-(d). Particularly, Theorem 1 (c) is a result from (3.2), (4.1) and (5.32). Here, $\chi_\Omega = 1$ in Ω, and 0 in Ω^C. □

6. Uniqueness

6.1. **Proof of Theorem 2.** Fix $\zeta_* := (\Psi, \varphi_-, p_-)$ and p_{ex} with satisfying Theorem 1 (i)-(iii), and suppose that $v_1, v_2 \in C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)$ satisfy

$$\mathcal{P}(\zeta_*, v_1) = p_{ex} = \mathcal{P}(\zeta_*, v_2). \tag{6.1}$$

If $\|v_j - v_c\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda} \leq \sigma_1$ in Theorem 1, then by (6.1) and Lemma 5.5, we have

$$v_1 - v_2 = -D_v\mathcal{P}^{-1}(\mathcal{P}(\zeta_*, v_1) - \mathcal{P}(\zeta_*, v_2) - D_v\mathcal{P}(v_1 - v_2)). \tag{6.2}$$

If $\mathcal{P}$ were continuously Fréchet differentiable, (6.2) would imply

$$\|v_1 - v_2\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda} \leq C\sigma_1 \|v_1 - v_2\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda}$$

so that if σ_1 is sufficiently small, then $v_1 = v_2$ holds, which provides Theorem 2 by Proposition 3.1 and 4.1. But it is unclear whether $\mathcal{P}$ is differentiable at other points near $\zeta_0 = (Id, \varphi_0^-, p_0^-, v_c)$ for the following reason.

For $v_j (j = 1,2)$ chosen above, let W_j^* be defined as in (4.9) associated with $\zeta_* = (\Psi, \varphi_-, p_-)$ and v_j, and let $E_j \in C^{1,\alpha}_{(-\alpha,\Gamma_w)}(\mathcal{N}^+_{r_s})$ be the unique solution to (4.8) with (4.10). Then, $E_1 - E_2$ satisfies $W_1^* \cdot D(E_1 - E_2) = (W_2^* - W_1^*) \cdot D(E_2 - E_0^+)$ in $\mathcal{N}^+_{r_s}$ with $D = (\partial_r, \partial_{\vartheta_1}, \cdots, \partial_{\vartheta_{n-1}})$ by

Lemma 4.4. Then, by the method of characteristics used in section 4.1,for any $\vartheta\in\Lambda$, E_1-E_2 has an expression of

$$\begin{aligned}&(E_1-E_2)(r_1,\vartheta)\\&=(E_1-E_2)|_{X(2r_1-r_s;r_1,\vartheta)}-\underbrace{\int_{r_1}^{2r_1-r_s}(W_2^*-W_1^*)\cdot D(E_2-E_0^+)|_{X(t;r_1,\vartheta)}dt}_{=:\mathcal{I}_*}\end{aligned}\tag{6.3}$$

where $X(t;r,\vartheta)$ solves (4.16) with $W=W_1^*$.

For $\mathcal{Q}$ in (5.3) to be Fréchet differentiable in sense of Lemma 5.2, we need to obtain a uniform bound of $\frac{\|\mathcal{I}_*\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda}}{\|v_1-v_2\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda}}$ for all v_1 sufficiently close to v_2 with $v_1\neq v_2$, but we only know, by Lemma 4.5, $D(E-E_0^+)\in C^{\alpha}_{(1-\alpha,\Gamma_w)}(\mathcal{N}^+_{r_s})$.

But still, by (4.15) and (4.16), it is likely that we have

$$\|\mathcal{I}_*\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda}\le C\|v_1-v_2\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda}$$

for a constant C depending on the data in sense of Remark 2.7. Then (6.2) may imply $\|v_1-v_2\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda}\le C\sigma_1\|v_1-v_2\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda}$, and so $v_1=v_2$ hold by reducing $\sigma_1>0$ in Theorem 1 if necessary. For that reason, we will prove the following lemma in this section.

Lemma 6.1. *For $\alpha\in(\frac12,1)$, $D_v\mathcal{P}$ defined in Lemma 5.2 is a bounded linear mapping from $C^{\alpha}_{(1-\alpha,\partial\Lambda)}(\Lambda)$ to itself and that $D_v\mathcal{P}$ is invertible in $C^{\alpha}_{(1-\alpha,\partial\Lambda)}(\Lambda)$.*

Lemma 6.2. *Fix $\alpha\in(\frac12,1)$. Fix $\zeta_*=(\Psi,\varphi_-,p_-)$ and $v_1,v_2\in C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)$ with ς_1,ς_2, defined in Theorem 1 and $\varsigma_4^{(j)}:=\|v_j-v_c\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda}$. Then there is a constant $C,\sigma_*>0$ depending on the data in sense of Remark 2.7 so that whenever $0<\varsigma_1,\varsigma_2,\varsigma_4^{(1)},\varsigma_4^{(2)}\le\sigma_*$ are satisfied there holds*

$$\begin{aligned}&\|\mathcal{P}(\zeta_*,v_1)-\mathcal{P}(\zeta_*,v_2)-D_v\mathcal{P}(v_1-v_2)\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda}\\&\le C(\varsigma_1+\varsigma_2+\sum_{j=1}^{2}\|v_j-v_c\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda})\|v_1-v_2\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda}.\end{aligned}\tag{6.4}$$

Before we prove Lemma 6.1 and Lemma 6.2, we assume that those lemmas hold true, and prove Theorem 2 first.

Proof of Theorem 2. Fix $\alpha\in(\frac12,1)$ and also fix $\Psi,\varphi_-,p_-,p_{ex}$ with $\varsigma_1,\varsigma_2,\varsigma_3$ in Theorem 1 satisfying $0<\varsigma_l\le\sigma$ for a constant $\sigma\in(0,\sigma_1]$ where σ_1 is as in Theorem 1.

Suppose that there are two functions $v_1,v_2\in C^{1,\alpha}_{(-\alpha,\partial\Lambda)}(\Lambda)$ satisfying (5.31), (5.32) with $v^*_{ex}=v_j$ for $j=1,2$. Then, by (5.32), (6.2), Lemma 6.1, Lemma 6.2, there holds

$$\|v_1-v_2\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda}\le C(\varsigma_1+\varsigma_2+\varsigma_3)\|v_1-v_2\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda}.\tag{6.5}$$

Since C depends on the data in sense of Remark 2.7, there exists a constant $\sigma_2\in(0,\sigma_1]$ with the same depndence as C so that we have

$$C(\varsigma_1+\varsigma_2+\varsigma_3)\le 3C\sigma_2<1.\tag{6.6}$$

Then, whenever $0<\sigma\le\sigma_2$, from (6.5) and (6.6), we conclude $v_1=v_2$. $\square$

The rest of section 6 is devoted to the proof of Lemma 6.1 and Lemma 6.2.

6.2. **Proof of Lemma 6.1.** Lemma 6.1 follows easily from:

Lemma 6.3. *Fix $\alpha \in (\frac{1}{2}, 1)$. For $F = (F_j)_{j=1}^n \in C^{\alpha}_{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}(\mathcal{N}^+_{r_s}, \mathbb{R}^n)$, and $g_1, g_3 \in C^{\alpha}_{(1-\alpha,\partial\Lambda)}(\Lambda)$, $g_2 \in C^{\alpha}_{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}(\Gamma_{w,r_s})$, the linear boundary problem*

$$\partial_j(a_{jk}(x,0)\partial_k\psi) = \partial_j F_j \quad in \ \mathcal{N}^+_{r_s} \tag{6.7}$$

$$D\psi\cdot\hat{r} - \mu_0\psi = g_1 \quad on \ S_0 \tag{6.8}$$

$$(a_{jk}(x,0)\partial_k\psi)\cdot\nu_w = g_2 \quad on \ \Gamma_{w,r_s} \tag{6.9}$$

$$(a_{jk}(x,0)\partial_k\psi)\cdot\hat{r} = g_3 \quad on \ \Gamma_{ex} \tag{6.10}$$

has a unique weak solution $\psi \in C^{1,\alpha}(\overline{\mathcal{N}^+_{r_s}} \setminus (\partial S_0 \cup \partial\Gamma_{ex})) \cap C^{\alpha}(\overline{\mathcal{N}^+_{r_s}})$. Moreover, there is a constant C depending only the data in sense of Remark 2.7 so that there holds

$$\|\psi\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,\alpha,\mathcal{N}^+_{r_s}} \le C(\|F\|^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{\alpha,\mathcal{N}^+_{r_s}} + \sum_{j=1,3}\|g_j\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda} + \|g_2\|^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{\alpha,\Gamma_{w,r_s}}). \tag{6.11}$$

The proof of Lemma 6.3 is given in Appendix B.

Proof of Lemma 6.1. By (5.7), for $D_v\mathcal{P}$ to be well-defined in $C^{\alpha}_{(1-\alpha,\partial\Lambda)}(\Lambda)$, the linear boundary problem (5.19), (5.20) must have a unique solution $\psi^{(w)} \in C^2(\mathcal{N}^+_{r_s}) \cap C^0(\overline{\mathcal{N}^+_{r_s}})$ for $w \in C^{\alpha}_{(1-\alpha,\partial\Lambda)}(\Lambda)$. This condition is satisfies by Lemma 6.3. Furthermore, (6.11) implies that $D_v\mathcal{P}$ maps $C^{\alpha}_{(1-\alpha,\partial\Lambda)}(\Lambda)$ into itself, and it is bounded in $C^{\alpha}_{(1-\alpha,\partial\Lambda)}(\Lambda)$ as well. Also, the mapping $w \mapsto a_2\mathcal{R}(\zeta_0)\psi^{(w)}(r_s,)$ is compact where $\mathcal{R}, a_2$ are defined in (5.3) and (5.8).

Suppose $D_v\mathcal{P}w_* = 0$ for some $w_* \in C^{\alpha}_{(1-\alpha,\partial\Lambda)}(\Lambda)$, then by (5.7), we have

$$w = -\frac{\mathcal{R}(\zeta_0)a_2}{\mathcal{Q}(\zeta_0)a_1}\psi^{(w)}(r_s,\cdot). \tag{6.12}$$

By (6.11), $\psi^{(w)}$ is in $C^{\alpha}(\overline{\mathcal{N}^+_{r_s}})$, then (6.12) implies $w_* \in C^{\alpha}(\overline{\Lambda})$, and again this provides $w_* = -\frac{\mathcal{R}(\zeta_0)a_2}{\mathcal{Q}(\zeta_0)a_1}\psi^{(w_*)}(r_s,\cdot) \in C^{1,\alpha}(\overline{\Lambda})$. This can be checked by arguments similar to the proof of Lemma 3.5. So one can follow the proof of Lemma 5.5 to show $w_* = 0$. By the Fredholm alternative theorem, we conclude that $D_v\mathcal{P}$ is invertible in $C^{\alpha}_{(1-\alpha,\partial\Lambda)}(\Lambda)$. □

6.3. **Proof of Lemma 6.2.** To prove Lemma 6.2, we need the following corollaries of Lemma 6.3. In this section, we fix Ψ, φ_-, p_- with ς_1, ς_2 defined in Theorem 1 satisfying $0 < \varsigma_l \le \sigma (l = 1, 2)$ for a constant $\sigma > 0$.

Corollary 6.4. *Fix $\alpha \in (\frac{1}{2}, 1)$. For $j = 1, 2$, let φ_j be the solution stated in Proposition 3.1 with $v_{ex} = v_j$ satisfying $\varsigma_4^{(j)} = \|v_j - v_c\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda} \le \sigma$ for $\sigma \in (0, \sigma_3]$, and with the transonic shock $r = f_j(x')$. Set $\phi_j := (\varphi_j - \varphi_0^+) \circ G^{-1}_{\varphi_j}$ for G_{φ_j} defined in (4.3). Then, there exist constant $C, \sigma^\flat > 0$ depending on the data in sense of Remark 2.7 so that whenever $0 < \sigma \le \sigma^\flat$ if $0 < \varsigma_1, \varsigma_2, \varsigma_4^{(j)} (j = 1, 2) \le \sigma$ are satisfied, then there hold*

$$\|\phi_1 - \phi_2\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,\alpha,\mathcal{N}^+_{r_s}} \le C\|v_1 - v_2\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda}, \tag{6.13}$$

$$\|f_1 - f_2\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda} \le C\|v_1 - v_2\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda}. \tag{6.14}$$

Proof. **Step 1.** For G_φ defined in (4.3), let us write $G_\varphi^{-1}(t,x') = (\omega^{(\varphi)}(t,x'),x')$ then we have

$$\upsilon^{(\varphi)}(\omega^{(\varphi)}(t,x'),x') = t \;\text{ for all }\; (t,x') \in [r_s,r_1]\times\overline{\Lambda}.$$

For convenience, we denote as υ_j,ω_j for $\upsilon^{(\varphi_j)},\omega^{(\varphi_j)} (j=1,2)$, then there holds

$$\upsilon_1(\omega_1(t,x'),x') - \upsilon_2(\omega_2(t,x'),x') = 0 \;\text{ for all }\; (t,x')\in[r_s,r_1]\times\overline{\Lambda} \tag{6.15}$$

By the definition of υ_j in (4.3), a direct computation shows that (6.15) is equivalent to

$$[q(r,x')]_{r=\omega_2(t,x')}^{\omega_1(t,x')} = -k(\phi_1-\phi_2)(t,x') + k[\phi_j(t,x')\chi(\omega_j(t,x'))]_{j=2}^1$$

where we set

$$\begin{aligned}
&q(r,x') := k(\varphi_0^-(r) - \varphi_0^+(r) + \psi_-(r,x'))(1-\chi(r)) + r\chi(r)\\
&\psi_-(r,x') := \varphi_-(r,x') - \varphi_0^-(r).
\end{aligned}$$

By the choice of k in (4.4), if ς_2 in Theorem 2.4 is sufficiently small, then there is a positive constant q_0 satisfying $\partial_r q \ge q_0 > 0$ for all $r\in[r_s,r_1]$ so that we can express $\omega_1-\omega_2$ as

$$(\omega_1-\omega_2)(t,x') = \frac{-k(\phi_1-\phi_2)(t,x') + k[\phi_j(t,x')\chi(\omega_j(t,x'))]_{j=2}^1}{\int_0^1 \partial_r q(a\omega_1(t,x') + (1-a)\omega_2(t,x'))da} \tag{6.16}$$

for $(t,x')\in[r_s,r_1]\times\overline{\Lambda}$. q_0 depends only on the data in sense of Remark 2.7. Furthermore, denoting as G_j, G_j^{-1} for $G_{\varphi_j}, G_{\varphi_j}^{-1} (j=1,2)$, since we have $D(G_j\circ G_j^{-1}) = DG_j^{-1}\circ DG_j(G_j^{-1}) = I_n$ for $j=1,2$, we obtain, in $\mathcal{N}_{r_s}^+$,

$$[DG_j(G_j^{-1})] = -DG_1(G_1^{-1})\circ(DG_1^{-1} - DG_2^{-1})\circ DG_2(G_2)^{-1}. \tag{6.17}$$

with setting $[DG_j(G_j^{-1})] := DG_1(G_1^{-1}) - DG_2(G_2^{-1})$.

Let us set $\Gamma_{cns} := \partial S_0 \cup \partial\Gamma_{ex}$. Combining (6.17) with (6.16), we obtain

$$\|\omega_1-\omega_2\|_{1,\alpha,\mathcal{N}_{r_s}^+}^{(-\alpha,\Gamma_{cns})} \le C(\|\phi_1-\phi_2\|_{1,\alpha,\mathcal{N}_{r_s}^+}^{(-\alpha,\Gamma_{cns})} + \|(\omega_1-\omega_2)\phi_2\|_{1,\alpha,\mathcal{N}_{r_s}^+}^{(-\alpha,\Gamma_{cns})}), \tag{6.18}$$

$$\|[DG_j(G_j^{-1})]\|_{\alpha,\mathcal{N}_{r_s}^+}^{(1-\alpha,\Gamma_{cns})} \le C(\|\phi_1-\phi_2\|_{1,\alpha,\mathcal{N}_{r_s}^+}^{(-\alpha,\Gamma_{cns})} + \|\phi_2(\omega_1-\omega_2)\|_{1,\alpha,\mathcal{N}_{r_s}^+}^{(-\alpha,\Gamma_{cns})}). \tag{6.19}$$

Step 2. For convenience, let us denote as $a_{kl}^{(j)}, F_k^{(j)}, b_1^{(j)}, \mu^{(j)}, g_m^{(j)}$ for $a_{kl}, F_k, b_1, \mu_f, g_m$ defined in (3.6), (3.7), (3.15) and (3.28) associated with $\Psi, \varphi_-, p_-, \varphi_j - \varphi_0^+, f_j$. Then, by (3.3)-(3.7), (3.8), (3.28) and (3.34), for each $j=1,2$, ϕ_j satisfies

$$\begin{aligned}
&\partial_k(\tilde{a}_{kl}(y,0)\partial_l\phi_j) = \partial_k\tilde{F}_k^{(j)} + \partial_k([\tilde{a}_{kl}(y,0) - \tilde{a}_{kl}^{(j)}]\partial_l\phi_j) =: \partial_k\mathfrak{F}_k^{(j)} \;\text{ in }\; \mathcal{N}_{r_s}^+,\\
&\partial_r\phi_j - \tilde{\mu}_0\phi_j = \tilde{g}_1^{(j)} + (\hat{r} - \tilde{b}_1^{(j)})\cdot D\phi_j - (\tilde{\mu}_0 - \tilde{\mu}^{f(j)})\phi_j =: \mathfrak{g}_1^{(j)} \;\text{ on }\; S_0\\
&(\tilde{a}_{kl}(y,0)\partial_l\phi_j)\cdot\nu_w = \tilde{g}_2^{(j)} + ([\tilde{a}_{kl}(y,0) - \tilde{a}_{kl}^{(j)}]\partial_l\phi_j)\cdot\nu_w =: \mathfrak{g}_2^{(j)} \;\text{ on }\; \Gamma_{w,r_s}\\
&(\tilde{a}_{kl}(y,0)\partial_l\phi_j)\cdot\hat{r} = (\upsilon_j - \upsilon_c) + ([\tilde{a}_{kl}(y,0) - \tilde{a}_{kl}^{(j)}]\partial_l\phi_j)\cdot\hat{r} =: \mathfrak{g}_3^{(j)} \;\text{ on }\; \Gamma_{ex}
\end{aligned} \tag{6.20}$$

where $\tilde{a}_{kl}(y,0), \tilde{\mu}_0$ are obtained from (6.7)-(6.10) through the change of variables $(r,x')\mapsto G_{\varphi_0^+}(r,x')$, and where $\tilde{a}_{kl}^{(j)}, \tilde{b}_1^{(j)}, \tilde{\mu}^{(j)}, \tilde{F}_k^{(j)}, \tilde{g}_m^{(j)}$ for $k,l\in\{1,\cdots,n\}$, $m\in\{1,2\}$ are obtained from $a_{kl}^{(j)}, b_1^{(j)}, \mu^{(j)}, F_k^{(j)}, g_m^{(j)}$ through the change of variables $(r,x')\mapsto G_j(r,x')$, and thus they smoothly depend on $D\Psi, (\rho_-,p_-,\varphi_-)$, $D\phi_j$ and $DG_j(G_j^{-1})$.

By (3.8), (3.28), (3.34), and by the definitions of $\mathfrak{F}_k^{(j)}, \mathfrak{g}_m^{(j)}$ in (6.20), one can directly check

$$\begin{aligned}
&\|\mathfrak{F}^{(1)}-\mathfrak{F}^{(2)}\|^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{\alpha,\mathcal{N}^+_{r_s}} \le C\kappa_*\delta(\phi_1,\phi_2)\\
&\|\mathfrak{g}_1^{(1)}-\mathfrak{g}_1^{(2)}\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda} \le C\kappa_*\delta(\phi_1,\phi_2)\\
&\|\mathfrak{g}_2^{(1)}-\mathfrak{g}_2^{(2)}\|^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{\alpha,\Gamma_{w,r_s}} \le C\kappa_*\delta(\phi_1,\phi_2)\\
&\|\mathfrak{g}_3^{(1)}-\mathfrak{g}_3^{(2)}\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda} \le C(\|v_1-v_2\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda}+\kappa_*)\delta(\phi_1,\phi_2)
\end{aligned} \tag{6.21}$$

where we set

$$\begin{aligned}
&\kappa_*:=\varsigma_1+\varsigma_2+\sum_{j=1}^2\|\phi_j\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,\alpha,\mathcal{N}^+_{r_s}},\\
&\delta(\phi_1,\phi_2):=\|\phi_1-\phi_2\|^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{\alpha,\mathcal{N}^+_{r_s}}\\
&\qquad+\|DG_1(G_1^{-1})-DG_2(G_2^{-1})\|^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{\alpha,\mathcal{N}^+_{r_s}}+\|G_1^{-1}-G_2^{-1}\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,\alpha,\mathcal{N}^+_{r_s}}.
\end{aligned}$$

Moreover, (6.18) and (6.19) imply

$$\delta(\phi_1,\phi_2)\le C(\|\phi_1-\phi_2\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,\alpha,\mathcal{N}^+_{r_s}}+\|(\omega_1-\omega_2)\phi_2\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,\alpha,\mathcal{N}^+_{r_s}}). \tag{6.22}$$

By Proposition 3.1 and (4.3), each ϕ_j satisfies

$$\|\phi_j\|^{(-1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{2,\alpha,\mathcal{N}^+_{r_s}}\le C(\varsigma_1+\varsigma_2+\|v_j-v_c\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda}) \tag{6.23}$$

for a constant C depending on the data in sense of Remark 2.7. By (6.18), (6.22) and (6.23), there exists a constant $\sigma^\flat\in(0,\sigma_3]$ so that whenever $0<\varsigma_1,\varsigma_2,\varsigma_4^{(j)}\le\sigma^\flat$ are satisfied, there holds

$$\delta(\phi_1,\phi_2)\le C\|\phi_1-\phi_2\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,\alpha,\mathcal{N}^+_{r_s}}. \tag{6.24}$$

Here, the constants C and $\sigma^\flat$ depend only on the data in sense of Remark 2.7. Then, applying Lemma 6.3 to $\phi_1-\phi_2$, by (6.20), (6.21) and (6.24), we obtain (6.13) reducing $\sigma^\flat$ if necessary.
Step 3. By arguing similarly as (2.47), and by the definition of ϕ_j, setting $\psi_-=\varphi_--\varphi_0^-$, we have

$$(f_1-f_2)(\vartheta)=\frac{[\phi_j(r_s,\vartheta)-\psi_-(f_j(\vartheta),\vartheta)]_{j=2}^1}{\int_0^1\partial_r(\varphi_0^--\varphi_0^+)(af_1(\vartheta)+(1-a)f_2(\vartheta))da} \tag{6.25}$$

for $x\in\Lambda$. By (6.25) and (6.23), we obtain

$$\|f_1-f_2\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda}\le C(\|\phi_1-\phi_2\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,\alpha,\mathcal{N}^+_{r_s}}+(\varsigma_2+\varsigma_4^{(1)}+\varsigma_4^{(2)})\|f_1-f_2\|^{(-\alpha,\partial\Lambda)}_{1,\alpha,\Lambda}). \tag{6.26}$$

We reduce $\sigma^\flat$ further to obtain (6.14) by (6.37) and (6.26). □

For any given $\alpha\in(\frac12,1)$, let $\psi\in C^2(\mathcal{N}^+_{r_s})\cap C^\alpha(\overline{\mathcal{N}^+_{r_s}})$ be be a unique solution to (6.20) with $\mathfrak{F}_k=\mathfrak{g}_1=\mathfrak{g}_2=0$ and $\mathfrak{g}_3=v_1-v_2$.

Corollary 6.5. *Fix $\alpha\in(\frac12,1)$. Let ϕ_1 and ϕ_2 be as in Corollary 6.4, Then, there are constants $C,\sigma^\natural>0$ depending only on the data in sense of Remark 2.7 so that whenever $0<\varsigma_1,\varsigma_2,\varsigma_4^{(1)},\varsigma_4^{(2)}\le\sigma^\natural$ are satisfied, then there holds,*

$$\|\phi_1-\phi_2-\psi\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,\alpha,S_0}\le C(\varsigma_1+\varsigma_2+\varsigma_4^{(1)}+\varsigma_4^{(2)})\|v_1-v_2\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda}. \tag{6.27}$$

Proof. By (6.20), and by the definition of ψ, $\psi_* := \phi_1 - \phi_2 - \psi$ satisfies So $\phi_1 - \phi_2 - u$ satisfies

$$\begin{aligned}
&\partial_k(\tilde{a}_{kl}(y,0)\partial_l\psi_*) = \partial_k(\mathfrak{F}^{(1)} - \mathfrak{F}^{(2)}) \text{ in } \mathcal{N}^+_{r_s}\\
&\partial_r\psi_* - \tilde{\mu}_0\psi_* = \mathfrak{g}_1^{(1)} - \mathfrak{g}_1^{(2)} \text{ on } S_0\\
&(\tilde{a}_{kl}(y,0)\partial_l\psi_*)\cdot\nu_w = \mathfrak{g}_2^{(1)} - \mathfrak{g}_2^{(2)} \text{ on } \Gamma_{w,r_s}\\
&(\tilde{a}_{kl}(y,0)\partial_l\psi_*)\cdot\hat{r} = \mathfrak{g}_3^{(1)} - \mathfrak{g}_3^{(2)} - (v_1 - v_2) \text{ on } \Gamma_{ex}.
\end{aligned}$$

By the definition of $\mathfrak{g}_3^{(j)}$ in (6.20), we emphasize that $\mathfrak{g}_3^* := \mathfrak{g}_3^{(1)} - \mathfrak{g}_3^{(2)} - (v_1 - v_2)$ satisfies

$$\|\mathfrak{g}_3^*\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda} \le C(\varsigma_1 + \varsigma_2 + \varsigma_4^{(1)} + \varsigma_4^{(2)})\delta(\phi_1,\phi_2)$$

where $\delta(\phi_1,\phi_2)$ is defined in (6.22). Then, by Lemma 6.3, Corollary 6.4 and (6.21), we conclude that if $\sigma^\natural$ is chosen sufficiently small depending on the data in sense of Remark 2.7, then $\psi_* = \phi_1 - \phi_2 - \psi$ satisfies

$$\|\psi_*\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,\alpha,\mathcal{N}^+_{r_s}} \le C(\varsigma_1 + \varsigma_2 + \varsigma_4^{(1)} + \varsigma_4^{(2)})\|v_1 - v_2\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda} \tag{6.28}$$

for a constant C depending only on the data in sense of Remark 2.7. □

Fix (Ψ,φ_-,p_-) in $B^{(1)}_\sigma(Id,\varphi_0^-,p_0^-)$. For simplicity, let us denote as $\mathcal{P}(v)$ for $\mathcal{P}(\Psi,\varphi_-,p_-,v)$
.

Proof of Lemma 6.2. **Step 1.**For convenience, for each $j = 1,2$, let us denote as $\mathcal{P}_j, \mathcal{Q}_j$ and $\mathcal{R}_j$ for $\mathcal{P}(\zeta_*,v_j), \mathcal{Q}(\zeta_*,v_j)$ and $\mathcal{R}(\zeta_*,v_j)$ respectively where $\mathcal{Q},\mathcal{R}$ are defined in (5.3). Then we can write

$$\mathcal{P}_1 - \mathcal{P}_2 - D_v\mathcal{P}(v_1 - v_2) = l_1\mathcal{R}_1 + l_2\mathcal{Q}_2 + l_3 \tag{6.29}$$

with

$$\begin{aligned}
&l_1 = \mathcal{Q}_1 - \mathcal{Q}_2 - D_v\mathcal{Q}(v_1 - v_2), \quad l_2 = \mathcal{R}_1 - \mathcal{R}_2 - D_v\mathcal{R}(v_1 - v_2),\\
&l_3 = (\mathcal{Q}_2 - \mathcal{Q}_0)D_v\mathcal{R}(v_1 - v_2) + (\mathcal{R}_1 - \mathcal{R}_0)D_v\mathcal{Q}(v_1 - v_2),
\end{aligned}$$

where we denote $\mathcal{Q}_0 = \mathcal{Q}(Id,\varphi_0^-,p_0^-,v_c)$, $\mathcal{R}_0 = \mathcal{R}(Id,\varphi_0^-,p_0^-,v_c)$. We estimate $l_1\mathcal{R}_1, l_2\mathcal{Q}_2, l_3$ separately.

By Lemma 6.1, Proposition 3.1, (4.14) and (4.15), we obtain

$$\|l_3\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda} \le C(\varsigma_1 + \varsigma_2 + \varsigma_4^{(1)} + \varsigma_4^{(2)})\|v_1 - v_2\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda}. \tag{6.30}$$

Step 2. Let $\psi \in C^2(\mathcal{N}^+_{r_s}) \cap C^\alpha(\overline{\mathcal{N}^+_{r_s}})$ be a unique solution to (5.19),(5.20) with $w = v_1 - v_2$, then by (6.3) and the definition of $D_v\mathcal{Q}$ in (5.6), we have

$$l_1(\vartheta) = (E_1 - E_2 - a_2\psi)|_{X(2r_1 - r_s; r_1,\vartheta)} + a_2(\psi|_{X(2r_1 - r_s; r_1,\vartheta)} - \psi(r_s,\vartheta)) + I_*(\vartheta) \tag{6.31}$$

where a_2, I_* are defined in (5.8), (6.3) respectively. First of all, by (4.14), (4.15), Corollary 6.4 and (6.18), we obtain

$$\|I_*\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda} \le C(\varsigma_1 + \varsigma_2 + \varsigma_4^{(2)})\|v_1 - v_2\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda}. \tag{6.32}$$

Secondly, for V_0 defined in Lemma 4.3, let us set $W_0^* := \frac{V_0}{V_0\cdot\hat{r}}$ so that we have $W_0^* = (1,0,\cdots,0)$ in (r,ϑ) coordinates which is a spherical coordinate system we specified in section 4. Then, by Lemma 4.3, there holds

$$\|W_1^* - W_0^*\|_{1,\alpha,\mathcal{N}_{r_s}^+}^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})} \le C(\varsigma+\varsigma_2+\varsigma_4^{(2)}) \tag{6.33}$$

for a constant C depending on the data in sense of Remark 2.7 where W_1^* is defined in the paragraph of (6.3). Thus, by (6.33) and Lemma 6.3, we obtain

$$\begin{aligned}&\|\psi|_{X_1(2r_1-r-s;r_1,\cdot)} - \psi(r_s,\cdot)\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)}\\ &\le C\|D\psi\|_{\alpha,\mathcal{N}_{e_s}^+}^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}\|W_1^*-W_0^*\|_{1,\alpha,\mathcal{N}_{r_s}^+}^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}\\ &\le C(\varsigma_1+\varsigma_2+\varsigma_4^{(1)})\|v_1-v_2\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)}\end{aligned} \tag{6.34}$$

for a constant C depending only on the data.

To complete step 2, it remains to estimate $\|(E_1-E_2-a_2\psi)(r_s,\cdot)\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)}$ because $X_1(2r_1-r_s;r_1,\vartheta)$ lies on $S_0=\{r=r_s\}\cap\mathcal{N}$ for all $\vartheta\in\Lambda$.

By the definition of a_2 in (5.8), $a_2\psi$ is expressed as

$$a_2\psi(r_s,\cdot) = a_2^{(1)}\psi(r_s,\cdot) + a_2^{(2)}\psi(r_s,\cdot) \tag{6.35}$$

with

$$a_2^{(1)} = \frac{1}{(B_0-\frac{1}{2}(\partial_r\varphi_0^+(r_s))^2)^{\frac{\gamma}{\gamma-1}}}\frac{\partial_r(p_{s,0}-p_0^+)(r_s)}{\partial_r(\varphi_0^--\varphi_0^+)(r_s)},\quad a_2^{(2)} = p_0^+(r_s)a_3 \tag{6.36}$$

where a_3 is defined in (5.23). So, we need to decompose E_1-E_2 into two parts so that one is comparable to $a_2^{(1)}\psi$, and the other is comparable to $a_2^{(2)}\psi$.

For each $j=1,2$, by (4.10), we have

$$E_j(r_s,\vartheta) = \frac{\rho_-(M_j\nabla\varphi_-\cdot\nu_s^{(j)})^2 + p_- - \rho_-K_s^{(j)}}{(B_0-\frac{1}{2}|D\Psi^{-1}(\Psi)\nabla\varphi_j|^2)^{\frac{\gamma}{\gamma-1}}} =: \frac{p_s^{(j)}}{U_j} \tag{6.37}$$

with

$$\begin{aligned}M_j &= \frac{|\nabla(\varphi_--\varphi_j)|(D\Psi^{-1})^T(D\Psi^{-1})}{|D\Psi^{-1}\nabla(\varphi_--\varphi_j)|}\\ K_s^{(j)} &= \frac{2(\gamma-1)}{\gamma+1}(\frac{1}{2}(M_j\nabla\varphi_-\cdot\nu_s^{(j)})^2 + \frac{\gamma p_-}{(\gamma-1)\rho_-})\end{aligned}$$

where every quantity is evaluated at $(f_j(\vartheta),\vartheta)$. Here, φ_j and f_j are as in Corollary 6.4, and $\nu_s^{(j)}$ is the unit normal on $S_j=\{r=f_j(\vartheta)\}$ toward the subsonic domain $\mathcal{N}_{f_j}^+$ of φ_j for $j=1,2$.

Set $U_0(r) := (B_0-\frac{1}{2}|\partial_r\varphi_0^+(r)|^2)^{\frac{\gamma}{\gamma-1}}$. Then, for each $j=1,2$, we can write $E_j(r_s,\vartheta) - E_0^+(f_j(\vartheta)) = \frac{p_s^{(j)}}{U_j} - \frac{p_{s,0}}{U_0} + \frac{p_0^+-p_{s,0}}{U_0}$ where $p_{s,0}, E_0^+$ is defined in (2.33) and Lemma 4.4 respectively, and where all the quantities are evaluated at $(f_j(\vartheta),\vartheta)$. Then, by Lemma 4.4, we can express $(E_1-E_2)(r_s,\vartheta)$ as

$$(E_1-E_2)(r_s,\vartheta) = z_1+z_2, \tag{6.38}$$

$$z_1 = [\frac{p_s^{(j)}}{U_j} - \frac{p_{s,0}(f_j(\vartheta))}{U_0(f_j(\vartheta))}]_{j=2}^1,\quad z_2 = [\frac{(p_{s,0}-p_0^+)(f_j(\vartheta))}{U_0(f_j(\vartheta))}]_{j=2}^1. \tag{6.39}$$

Here, $[F_j]_{j=2}^1$ is defined by $[F_j]_{j=2}^1 := F_1 - F_2$.

We claim that, for a constant C depending on the data in sense of Remark 2.7, there hold

$$\|z_1 - a_2^{(2)}\psi(r_s,\cdot)\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)} \le C\varsigma_*\|v_1-v_2\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)}, \tag{6.40}$$

$$\|z_2 - a_2^{(1)}\psi(r_s,\cdot)\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)} \le C\varsigma_*\|v_1-v_2\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)}. \tag{6.41}$$

where we set $\varsigma_* := \varsigma_1+\varsigma_2+\varsigma_4^{(1)}+\varsigma_4^{(2)}$.

Step 3.(Verification of (6.41)**)** We rewrite z_2 in (6.39) as

$$\begin{aligned} &z_2(\vartheta)\\ &= \frac{[(p_{s,0}-p_0^+)|_{r=f_j(\vartheta)}]_{j=2}^1}{U_0(r_s)} + [(p_{s,0}-p_0^+)|_{r=f_j(\vartheta)}(\frac{1}{U_0(f_j(\vartheta))}-\frac{1}{U_0(r_s)})]_{j=2}^1\\ &=: z_2^{(1)}+z_2^{(2)}. \end{aligned} \tag{6.42}$$

By (2.33), Proposition 3.1 and Corollary (6.4), we obtain

$$\|z_2^{(2)}\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)} \le C\varsigma_*\|v_1-v_2\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)}. \tag{6.43}$$

By (6.25), we can write

$$z_2^{(1)}(\vartheta) - a_2^{(1)}\psi(r_s,\vartheta) = \frac{\tilde a_2^{(1)}(\vartheta)[\phi_j(r_s,\vartheta)-\psi_-(f_j(\vartheta),\vartheta)]_{j=2}^1 - a_2^{(1)}\psi_(r_s,\vartheta)}{U_0(r_s)}$$

with

$$\tilde a_2^{(1)}(\vartheta) = \frac{\int_0^1 \partial_r(p_{s,0}-p_0^+)(af_1(\vartheta)+(1-a)f_2(\vartheta))da}{U_0(r_s)\int_0^1\partial_r(\varphi_0^- - \varphi_0^+)(af_1(\vartheta)+(1-a)f_2(\vartheta))da}.$$

where each ϕ_j is as in Corollary 6.4, and where we denote as $\psi_- = \varphi_- - \varphi_0^-$. By Corollary 6.4 and Corollary 6.5, there holds

$$\|z_2^{(1)} - a_2^{(1)}\psi(r_s,\cdot)\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)} \le C\varsigma_*\|v_1-v_2\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)}. \tag{6.44}$$

The constants C in (6.43), (6.44) depend only on the data in sense of Remark 2.7. Then, (6.41) is obtained by (6.43) and (6.44).

Step 4. (Verification of (6.40)**)** By (5.19) and (5.22), it is easy to see that for $a_2^{(2)}\psi(r_s,\cdot)$ in (6.35), there holds

$$a_2^{(2)}\psi(r_s,\cdot) = \frac{\gamma p_0^+(r_s)\partial_r\varphi_0^+(r_s)}{(\gamma-1)(B_0-\frac12(\partial_r\varphi_0^+(r_s))^2)^{\frac{2\gamma-1}{\gamma-1}}}\partial_r\psi(r_s,\cdot) = a_4\partial_r\psi(r_s,\cdot). \tag{6.45}$$

Similarly to step 3 above, we rewrite z_1 in (6.39) as

$$\begin{aligned} z_1 &= p_0^+(r_s)[\frac{1}{U_j^*}-\frac{1}{U_0(f_j(\vartheta))}]_{j=2}^1 + [(\frac{p_s^{(j)}}{U_j}-\frac{p_0^+}{U_j^*}) + \frac{p_0^+(r_s)-p_{s,0}(f_j(\vartheta))}{U_0(f_j(\vartheta))}]_{j=2}^1\\ &=: z_1^{(1)}+z_1^{(2)} \end{aligned} \tag{6.46}$$

where we set

$$U_j^* := (B_0 - \frac12|\nabla\varphi(f_j(\vartheta),\vartheta)|^2)^{\frac{\gamma}{\gamma-1}}.$$

By (2.33), Proposition 3.1, (6.37) and Corollary 6.4, $z_1^{(2)}$ in (6.46) satisfies

$$\|z_1^{(2)}\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)} \le C\varsigma_* \|v_1 - v_2\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)}. \tag{6.47}$$

For each $j = 1, 2$, we can write

$$\begin{aligned} & p_0^+ [\frac{1}{U_j^*} - \frac{1}{U_0(f_j(\vartheta))}] \\ &= \int_0^1 \frac{\gamma p_0^+(r_s)\beta_4^{(j)}(\vartheta)\cdot\nabla(\varphi_j - \varphi_0^+)|_{(f_j(\vartheta),\vartheta)}}{(\gamma-1)(B_0 - \frac{1}{2}(|\partial_r\varphi_0^+(f_j(\vartheta))|^2 + 2t\beta_4^{(j)}\cdot\nabla(\varphi_j - \varphi_0^+)|_{(f_j(\vartheta),\vartheta)}))^{\frac{2\gamma-1}{\gamma-1}}} dt \\ &= a_4^{(j)} \cdot \nabla(\varphi_j - \varphi_0^+)|_{(f_j(\vartheta),\vartheta)} \end{aligned} \tag{6.48}$$

with $\beta_4^{(j)}(\vartheta) = \partial_r\varphi_0^+(f_j(\vartheta))\hat{r} + \frac{1}{2}\nabla(\varphi_j - \varphi_0^+)|_{(f_j(\vartheta),\vartheta)}$.

By Proposition 3.1, we have, for a constant C depending on the data in sense of Remark 2.7

$$\|a_4^{(j)} - a_4\hat{r}\|_{1,\alpha,\Lambda}^{(-\alpha,\partial\Lambda)} \le C\varsigma_*. \tag{6.49}$$

So, if we write $z_1^{(1)} - a_4\partial_r\psi(r_s,\cdot)$ as

$$\begin{aligned} & z_1^{(1)} - a_4\partial_r\psi(r_s,\vartheta) \\ &= a_4([\partial_r(\varphi_j - \varphi_0^+)|_{(f_j(\vartheta,\vartheta))}]_{j=2}^1 - \psi(r_s,\vartheta)) + [(a_4^{(j)} - a_4)\cdot\nabla(\varphi_j - \varphi_0^+)|_{(f_j(\vartheta),\vartheta)}]_{j=2}^1, \end{aligned}$$

then, by Proposition 3.1, (4.3), Corollary 6.4, Corollary 6.5, (6.49), there is a constant C depending on the data in sense of Remark 2.7 so that there holds

$$\|z_1^{(1)} - a_4\partial_r\psi(r_s,\vartheta)\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)} \le C\varsigma_*\|v_1 - v_2\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)}. \tag{6.50}$$

Thus (6.40) holds true. Finally, (6.40) and (6.41) imply, for a constant C depending on the data in sense of Remark 2.7,

$$\|\mathcal{R}_1 l_1\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)} \le C\varsigma_*\|v_1 - v_2\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)}. \tag{6.51}$$

Similarly, one can also show

$$\|\mathcal{Q}_2 l_2\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)} \le C\varsigma_*\|v_1 - v_2\|_{\alpha,\Lambda}^{(1-\alpha,\partial\Lambda)}. \tag{6.52}$$

As (6.52) can be checked more simply than (6.51), we omit the details for the verification of (6.52).

Finally, combining (6.30), (6.51) and (6.52) all together, we obtain (6.4), and this competes the proof of Lemma 6.2.

□

Appendix A. Proof of Proposition 5.1

Before we prove Proposition 5.1, we first prove the following lemma.

Lemma A.1 (Right Inverse Mapping Theorem)**.** *Let $\mathfrak{C}_1$ be a compactly imbedded subspace of a Banach space $\mathfrak{B}_1$. Also, suppose that $\mathfrak{C}_2$ is a subspace of another Banach space $\mathfrak{B}_2$. For a given point $(x_0, y_0) \in \mathfrak{C}_1 \times \mathfrak{C}_2$, suppose that a mapping $\mathcal{H}$ satisfies the following conditions:*

(i) *$\mathcal{H}$ maps a neighborhood of x_0 in $\mathfrak{B}_1$ to $\mathfrak{B}_2$, and maps a neighborhood of x_0 in $\mathfrak{C}_1$ to $\mathfrak{C}_2$, with $\mathcal{H}(x_0) = y_0$,*

(ii) *whenever a sequence* $\{x_k\} \subset \mathfrak{C}_1$ *near* x_0 *converges to* x_* *in* $\mathfrak{B}_1$*, the sequence* $\{\mathcal{H}(x_k)\} \subset \mathfrak{C}_2$ *converges to* $\mathcal{H}(x_*)$ *in* $\mathfrak{B}_2$*,*

(iii) $\mathcal{H}$*,as a mapping from* $\mathfrak{B}_1$ *to* $\mathfrak{B}_2$*, also as a mapping from* $\mathfrak{C}_1$ *to* $\mathfrak{C}_2$*, is Fréchet differentiable at* x_0*,*

(iv) *the Fréchet derivative* $D\mathcal{H}(x_0)$*, as a mapping from* $\mathfrak{B}_1$ *to* $\mathfrak{B}_2$*, also as a mapping from* $\mathfrak{C}_1$ *to* $\mathfrak{C}_2$*, is invertible.*

Then there is a small neighborhood $\mathcal{V}(x_0)$ *of* x_0 *in* $\mathfrak{C}_1$*, and a small neighborhood* $\mathcal{W}(y_0)$ *of* y_0 *in* $\mathfrak{C}_2$ *so that* $\mathcal{H} : \mathcal{V}(x_0) \to \mathcal{W}(y_0)$ *has a right inverse i.e., there is a mapping* $\mathcal{H}^{-1}_{Right} : \mathcal{W}(y_0) \to \mathcal{V}(x_0)$ *satisfying*

$$\mathcal{H} \circ \mathcal{H}^{-1}_{Right} = id \tag{A.1}$$

and moreover $\mathcal{H}^{-1}_{Right}$ *satisfies* $\mathcal{H}^{-1}_{Right} \circ \mathcal{H}(x_0) = x_0$*.*

Proof. Denote $D\mathcal{H}(x_0)$ as $D\mathcal{H}_0$. For $t \in \mathbb{R}_+$, and $v, w \in \mathfrak{C}_1$, define

$$\mathcal{H}^*(t, v, w) := \frac{1}{t}\big(\mathcal{H}(x_0 + t(v + w)) - \mathcal{H}(x_0)\big) - D\mathcal{H}_0 w.$$

For $r > 0$ and $a_0 \in \mathfrak{C}_1$, set $B_r(a_0) := \{y \in \mathfrak{C}_1 : \|y - a_0\|_{\mathfrak{C}_1} \le r\}$. For a fixed $(t, w) \in (0, \varepsilon_0] \times \partial B_1(0)$, we define

$$\mathcal{H}^*_{t,w}(v) := D\mathcal{H}_0^{-1}(\mathcal{H}^*)(t, v, w) - v.$$

If the constant ε_0 is sufficiently small then we have $\mathcal{H}^*_{t,w}$ maps $B_1(0)$ into itself. Since $B_1(0)$ is bounded in $\mathfrak{C}_1$, $B_1(0)$ is a compact convex subset of $\mathfrak{B}_1$. By Lemma A.1 (ii) and (iii), the Schauder fixed point theorem applies to $\mathcal{H}^*_{t,w}$. Thus $\mathcal{H}^*_{t,w}$ has a fixed point $v_* = v_*(t, w)$ satisfying

$$\mathcal{H}^*_{t,w}(v_*) = v_*.$$

By the definition of $\mathcal{H}^*_{t,w}$, v_* satisfies

$$\begin{aligned} \mathcal{H}^*_{t,w}(v_*) = v_* &\Leftrightarrow \mathcal{H}^*(t, v_*, w) = 0 \\ &\Leftrightarrow \mathcal{H}(x_0 + t(v_* + w)) = \mathcal{H}(x_0) + tD\mathcal{H}_0 w. \end{aligned} \tag{A.2}$$

We claim

$$\mathcal{H}^{-1}_{Right}(z) := x_0 + t(v_* + w) \tag{A.3}$$

for $z = \mathcal{H}(x_0) + tD\mathcal{H}_0 w$. Obviously, $\mathcal{H}^{-1}_{Right}(z)$ in (A.3) satisfies (A.1). So it remains to check if $\mathcal{H}^{-1}_{Right}$ in(A.3) is well-defined.

Set $\mathcal{W}(y_0) := \{y_0 + tD\mathcal{H}_0 w : t \in [0, \varepsilon_0], w \in \partial B_1(0)\}$. Since $D\mathcal{H}_0$ is invertible, $\mathcal{W}(y_0)$ covers a small neighborhood of y_0 in $\mathfrak{C}_2$. Suppose $y_0 + t_1 D\mathcal{H}_0 w_1 = y_0 + t_2 D\mathcal{H}_0 w_2$ for $t_1, t_2 \in [0, \varepsilon_0]$ and $w_1, w_2 \in \partial B_1(0)$. If $t_1 = 0$ then t_2 must be 0 as well. If $t_1 \ne 0$ then we have $D\mathcal{H}_0 w_1 = \frac{t_2}{t_1} D\mathcal{H}_0 w_2$ and this implies $w_1 = \frac{t_2}{t_1} w_2$. So we obtain $t_1 = t_2$ and $w_1 = w_2$.

By (A.3), we also have $\mathcal{H}^{-1}_{Right} \circ \mathcal{H}(x_0) = \mathcal{H}^{-1}_{Right}(\mathcal{H}(x_0) + 0w) = x_0$. $\square$

Remark A.2. *By* (A.3)*, there holds*

$$\|\mathcal{H}^{-1}(z) - x_0\|_{\mathfrak{C}_1} \le 2t = 2\frac{\|z - \mathcal{H}(x_0)\|_{\mathfrak{C}_2}}{\|D\mathcal{H}_0 w\|_{\mathfrak{C}_2}} \le 2\|D\mathcal{H}_0^{-1}\|\|z - \mathcal{H}(x_0)\|_{\mathfrak{C}_2}$$

where we set $\|D\mathcal{H}_0^{-1}\| := \sup_{\|v\|_{\mathfrak{C}_2} = 0} \|D\mathcal{H}_0^{-1}\|_{\mathfrak{C}_1}$*.*

Proposition 5.1 can be easily proven from Lemma A.1.

Proof of Proposition 5.1. Let us define $\mathcal{H}(x,y)$ by

$$\mathcal{H}(x,y) := (\mathcal{F}(x,y), y). \tag{A.4}$$

In Lemma A.1, replacing $\mathfrak{B}_1$ by $\mathfrak{B}_1 \times \mathfrak{B}_2$, $\mathfrak{C}_1$ by $\mathfrak{C}_1 \times \mathfrak{C}_2$, $\mathfrak{B}_2$ by $\mathfrak{B}_3$, $\mathfrak{C}_2$ by $\mathfrak{C}_3$, x_0 by (x_0, y_0) and y_0 by $(\mathbf{0}, y_0)$, $\mathcal{H}$ in (A.4) satisfies Lemma A.1 (i)-(iii). In particular, the Fréchet derivative of $\mathcal{H}$ at (x_0, y_0), which we write as $D\mathcal{H}_0$, is given by

$$D\mathcal{H}_0(x,y) = (D_x\mathcal{F}_0 x + D_y\mathcal{F}_0 y, y)$$

where we denote $D_{(x,y)}\mathcal{F}(x_0,y_0)$ as $(D_x\mathcal{F}_0, D_y\mathcal{F}_0)$. By Proposition 5.1 (iv), $\mathcal{H}$ satisfies Lemma A.1 (iv). Therefore, by Lemma A.1, $\mathcal{H}$ has a right inverse $\mathcal{H}^{-1}_{Right}$ in a small neighborhood of (x_0, y_0) in $\mathfrak{C}_1 \times \mathfrak{C}_2$. In particular, for $y \in \mathfrak{C}_2$ with $\|y - y_0\|_{\mathfrak{C}_2}$ being sufficiently small, $\mathcal{H}^{-1}_{Right}$ is well defined. Write $\mathcal{H}^{-1}_{Right}(0, y) = (x^*(y), y)$ then we obtain $\mathcal{F}(x^*(y), y) = 0$. □

Appendix B. Proof of Lemma 6.3

Proof of Lemma 6.3.

Step 1. Fix $F^* = (F_j)_{j=1}^n \in C^{\alpha}_{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}(\mathcal{N}^+_{r_s}, \mathbb{R}^n)$, and $g_1^*, g_3^* \in C^{\alpha}_{(1-\alpha,\partial\Lambda)}(\Lambda)$, $g_2^* \in C^{\alpha}_{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}(\Gamma_{w,r_s})$. Then, there are sequences $\{F^{(k)} = (F_j^{(k)})_{j=1}^n\} \subset C^{\alpha}(\overline{\mathcal{N}^+_{r_s}}, \mathbb{R}^n)$, and $\{g_1^{(k)}\}, \{g_3^{(k)}\} \subset C^{\alpha}(\overline{\Lambda})$, $\{g_2^{(k)}\} \subset C^{\alpha}(\overline{\Gamma_{w,r_s}})$ satisfying

$$\begin{aligned}&\lim_{k\to\infty} \|F^{(k)} - F^*\|^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{\alpha,\mathcal{N}^+_{r_s}} = \lim_{k\to\infty} \|g_1^{(k)} - g_1^*\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda}\\ &= \lim_{k\to\infty} \|g_2^{(k)} - g_2^*\|^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{\alpha,\Gamma_{w,r_s}} = \lim_{k\to\infty} \|g_3^{(k)} - g_3^*\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda} = 0.\end{aligned}$$

For each k, let $\psi^{(k)} \in C^2(\mathcal{N}^+_{r_s}) \cap C^0(\overline{\mathcal{N}^+_{r_s}})$ be the unique solution to (6.7)-(6.10) with $F = F^{(k)}, g_l = g_l^{(k)}$ for $l = 1,2,3$. The unique existence of $\psi^{(k)}$ can be proven by the method of continuity as in the proof of Lemma 3.5.

If (6.11) holds true for $F \in C^{\alpha}(\overline{\mathcal{N}^+_{r_s}}, \mathbb{R}^n), g_1, g_2 \in C^{\alpha}(\overline{\Lambda}), g_2 \in C^{\alpha}(\overline{\Gamma_{w,r_s}})$, then we have

$$\lim_{k,m\to\infty} \|\psi^{(k)} - \psi^{(m)}\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,\alpha,\mathcal{N}^+_{r_s}} = 0. \tag{B.1}$$

By (B.1), $\psi^{(k)}$ converges to a function ψ^* in C^2 for any compact subset of $\mathcal{N}^+_{r_s}$, so ψ^* satisfies (6.7) with $F = F^*$ in $\mathcal{N}^+_{r_s}$. Also, ψ^* satisfies (6.8)-(6.10) on the relative interior of each corresponding boundary with $g_l = g_l^*$ for $l = 1,2,3$. Since $\{\psi^{(k)}\}$ is a Cauchy sequence in $C^{1,\alpha}_{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}(\overline{\mathcal{N}^+_{r_s}})$, we have $\psi^* \in C^{1,\alpha}_{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}(\mathcal{N}^+_{r_s}) \subset C^{\alpha}(\overline{\mathcal{N}^+_{r_s}})$. Then, by [LI1, Corollary 2.5], $\mu_0 > 0$ in (6.8) implies that $\psi^* \in C^2(\mathcal{N}^+_{r_s}) \cap C^0(\overline{\mathcal{N}}^+_{r_s})$ is a unique solution to (6.7)-(6.10). Moreover, ψ^* satisfies (6.11) by (B.1). Thus, it suffices to prove (6.11) for

$$F \in C^{\alpha}(\overline{\mathcal{N}^+_{r_s}}, \mathbb{R}^n),\ g_1, g_2 \in C^{\alpha}(\overline{\Lambda}),\ g_2 \in C^{\alpha}(\overline{\Gamma_{w,r_s}}). \tag{B.2}$$

So, for the rest of proof, we assume (B.2).

Step 2. By the local scalings as in the proof of Lemma 3.5, one can easily check that if $\psi \in C^2(\mathcal{N}^+_{r_s}) \cap C^0(\overline{\mathcal{N}^+_{r_s}})$ solves (6.7)-(6.10), then there is a constant C depending on $\mathcal{N}^+_{r_s}$ and α so that there holds

$$\|\psi\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,\alpha,\mathcal{N}^+_{r_s}} \le C(|\psi|_{\alpha,\mathcal{N}^+_{r_s}} + K(\psi, F, g_1, g_2, g_3)) \tag{B.3}$$

where we set

$$K(\psi,F,g_1,g_2,g_3):=\|F\|^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{\alpha,\mathcal{N}^+_{r_s}}+\|g_2\|^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{\alpha,\Gamma_{w,r_s}}+\sum_{l=1,3}\|g_l\|^{(1-\alpha,\partial\Lambda)}_{\alpha,\Lambda}.$$

So it suffices to estimate $|\psi|_{0,\alpha,\mathcal{N}^+_{r_s}}$.

Since $\mathcal{N}^+_{r_s}$ is a cylindrical domain with the cross-section Λ in $(r,x')-$coordinates in (2.22), there is a constant $\kappa_0>0$ depending only on n,Λ to satisfy, for any $x_0\in\mathcal{N}^+_{r_s}$ and $R>0$,

$$\frac{1}{\kappa_0}\le\frac{vol(B_R(x_0)\cap\mathcal{N}^+_{r_s})}{R^n}\le\kappa_0.$$

So, if there is a constant $R_*>0$ and $M>0$ satisfying

$$\sup_{x_0\in\mathcal{N}^+_{r_s},R>0}\frac{1}{R^{n-2+2\alpha}}\int_{B_R(x_0)\cap\mathcal{N}^+_{r_s}}|D\psi|^2\le M^2\ \text{ for all }\ R\in(0,R_*],\tag{B.4}$$

then there holds

$$|\psi|_{\alpha,\mathcal{N}^+_{r_s}}\le C(\frac{1}{R_*^\alpha}|\psi|_{0,\mathcal{N}^+_{r_s}}+M)\tag{B.5}$$

assuming $R_*<1$ without loss of generality where the constant C depends only on n,α,Λ,r_s,r_1. So, we devote the rest of proof to find a constant $M>0$ satisfying (B.4).

To obtain (B.4), we need to consider the three cases: (i) $B_R(x_0)\subset\mathcal{N}^+_{r_s}$, (ii) $x_0\in\partial\mathcal{N}^+_{r_s}\setminus(\partial S_0\cup\partial\Gamma_{ex})$ and $B_R(x_0)\cap(\partial S_0\cup\partial\Gamma_{ex})=\emptyset$, (iii) $x_0\in\partial S_0\cup\partial\Gamma_{ex}$. More general cases can be treated to these three cases. Also, case (i), (ii) are easier to handle than case (iii). So we only consider case (iii).

Step 3. Assume $x_0\in\partial S_0$ because the case of $x_0\in\partial\Gamma_{ex}$ can be treated more simply. For a fixed point $x_0\in\partial S_0$, let $x_0=(r_s,x_0')$ in the (r,x')-coordinates given in (2.22). Then, there is a constant $R_0>0$ depending on Λ, and a smooth diffeomorphism h defined in a neighborhood $\mathcal{O}_{x_0'}$ of x_0' in $\overline{\Lambda}$ so that h flattens $\partial\Lambda$ near x_0', and moreover the followings hold:

$$\begin{aligned}&(a)\ \ h(x_0')=0\in\mathbb{R}^{n-1},\qquad (b)\ \text{ for any }\ x'\in\mathcal{O}_{x_0'}\cap\partial\Lambda,\ \ h(x')\in\mathbb{R}^{n-2}\times\{0\},\\ &(c)\ \text{ for any }\ (r,x')\in B_{R_0}(x_0)\cap\mathcal{N}^+_{r_s},\ \ (r,h(x'))\in(r_s,r_1)\times\mathbb{R}^{n-2}\times\mathbb{R}_+.\end{aligned}$$

For a constant $R>0$ and $y_0:=(r_s,h(x_0'))=(r_s,0)\in\mathbb{R}^n$, let us set

$$\begin{aligned}&\mathcal{D}_R:=B_R(y_0)\cap\{(r,h(x')):(r,x')\in B_{R_0}(x_0)\cap\mathcal{N}^+_{r_s}\},\\ &\Sigma_{\Gamma_w,R}:=B_R(y_0)\cap\{(r,h(x')):(r,x')\in B_{R_0}(x_0)\cap\Gamma_w\},\\ &\Sigma_{S_0,R}:=B_R(y_0)\cap\{(r,h(x')):(r,x')\in B_{R_0}(x_0)\cap S_0\}.\end{aligned}\tag{B.6}$$

For $(r,x')\in\mathcal{N}^+_{r_s}$, let us write $y=(r,y')=(r,h(x'))$, and $\phi(y)=\psi(r,x')$. By (6.7)-(6.9), ϕ satisfies

$$\begin{aligned}&\partial_k(\tilde a_{kl}(y,0)\partial_l\phi)=\partial_k\tilde F_k\ \text{ in }\ \mathcal{D}_R,\quad \partial_r\phi-\mu_0\phi=\tilde g_1\ \text{ on }\ \Sigma_{S_0,R}\\ &(\tilde a_{kl}(y,0)\partial_l\phi)\cdot\tilde\nu_w=\tilde g_2\ \text{ on }\ \Sigma_{\Gamma_w,R}\end{aligned}\tag{B.7}$$

where $\tilde a_{kl},\tilde F_k,\tilde g_1,\tilde g_2$ are obtained from a_{kl},F_k,g_1,g_2 in(6.7)-(6.10) through the change of variables $(r,x')\mapsto(r,h(x'))$. So $[\tilde a_{kl}]^n_{k,l=1}$ is strictly positive, and the regularity of $\tilde a_{kl},\tilde F_k,\tilde g_1,\tilde g_2$ are same as the regularity of a_{kl},F_k,g_1,g_2 in $\mathcal{D}_R$. Here, $\tilde\nu_w$ is the inward unit normal to $\Sigma_{\Gamma_w,R}$.

Since h is smooth, and Λ is compact, there is a constant $R_1 > 0$ depending on n, Λ so that we have

$$\partial\mathcal{D}_{2R_1} \setminus (\Sigma_{S_0,2R_1} \cup \Sigma_{\Gamma_w,2R_1}) \subset \partial B_{2R_1}(y_0).$$

Step 4. Fix $R \in (0, R_1]$, and set $\Sigma_{\partial B,R} := \partial\mathcal{D}_R \cap \partial B_R(y_0)$. Write ϕ in (B.7) as $\phi = u + w$ for a weak solution u to

$$\begin{aligned} &\partial_k(\tilde{a}_{kl}(y_0,0)\partial_l u) = 0 \text{ in } \mathcal{D}_R, \qquad u = \phi \text{ on } \Sigma_{\partial B,R} \\ &\partial_r u - \mu_0 u = 0 \text{ on } \Sigma_{S_0,R}, \qquad (\tilde{a}_{kl}(y_0,0)\partial_l u)\cdot\tilde{\nu}_w = 0 \text{ on } \Sigma_{\Gamma_w,R}. \end{aligned} \tag{B.8}$$

Such $u \in W^{1,2}(\mathcal{D}_R)$ uniquely exists by Lemma 2.5, and by a basic estimate for harmonic functions as in [HL, Lemma 3.10], there is a constant C depending on the data in sense of Remark 2.7 to satisfy

$$\int_{\mathcal{D}_{\varrho_1}} |Du|^2 dy \le C(\frac{\varrho_1}{\varrho_2})^n \int_{\mathcal{D}_{\varrho_2}} |Du|^2 dy \text{ for } 0 < \varrho_1 \le \varrho_2 \le R. \tag{B.9}$$

Step 5. Then, by (B.7) and (B.8), $w = \phi - u$ satisfies

$$\begin{aligned} &\partial_k(\tilde{a}_{kl}(y_0,0)\partial_l w) = \partial_k([\tilde{a}_{kl}(y_0,0) - \tilde{a}_{kl}(y,0)]\partial_l\phi + \tilde{F}_k) \text{ in } \mathcal{D}_R, \\ &\partial_r w - \mu_0 w = \tilde{g}_1 \text{ on } \Sigma_{S_0,R}, \\ &(\tilde{a}_{kl}(y_0,0)\partial_l w)\cdot\tilde{\nu}_w = ([\tilde{a}_{kl}(y_0,0) - \tilde{a}_{kl}(y,0)]\partial_l\phi)\cdot\tilde{\nu}_w + \tilde{g}_2 \text{ on } \Sigma_{\Gamma_w,R}, \\ &w = 0 \text{ on } \Sigma_{\partial B,R}. \end{aligned} \tag{B.10}$$

By (3.41), (3.52) and (B.10), there is a constant $\lambda, C > 0$ depending on the data in sense of Remark 2.7 so that there holds

$$\lambda\int_{\mathcal{D}_R} |Dw|^2\, dy \le C(\int_{\mathcal{D}_R} |h_1||Dw|dy + \int_{\Sigma_{S_0,R}} |h_2||w|\, dA_y + \int_{\Sigma_{\Gamma_w,R}} |h_3||w|dA_y) \tag{B.11}$$

where we set

$$\begin{aligned} &|h_1(y)| = \delta^*(y_0,y)|D\phi(y)| + |\tilde{F}(y)| \\ &|h_2(y)| = \delta^*(y_0,y)|D\phi(y)| + |\tilde{F}(y)| + |w(y)| + |\tilde{g}_1(y)| \\ &|h_3(y)| = \delta^*(y_0,y)|D\phi(y)| + |\tilde{F}(y)| + |\tilde{g}_2(y)| \end{aligned} \tag{B.12}$$

with $\delta^*(y_0,y) = (\sum_{k,l=1}^n |\tilde{a}_{kl}(y_0,0) - \tilde{a}_{kl}(y,0)|^2)^{1/2}$.

Before we proceed further, we first note that, by the definition of $\phi, \tilde{F} = (\tilde{F}_k)_{k=1}^n, \tilde{g}_1, \tilde{g}_2$ in (B.7), for any $\beta \in [0,\alpha]$, there is a constant $C(\beta,\Lambda) > 0$ depending on β, Λ so that we have

$$\begin{aligned} &\|\phi\|_{1,\beta.\mathcal{D}_R}^{(-\alpha,\overline{\Sigma_{S_0,R}}\cap\overline{\Sigma_{\Gamma_w,R}})} \le C(\beta,\Lambda)\|\psi\|_{1,\beta,\mathcal{N}_{r_s}^+}^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})} \\ &\|\tilde{F}\|_{\beta.\mathcal{D}_R}^{(1-\alpha,\overline{\Sigma_{S_0,R}}\cap\overline{\Sigma_{\Gamma_w,R}})} \le C(\beta,\Lambda)\|F\|_{\beta,\mathcal{N}_{r_s}^+}^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})} \text{ with } F = (F_k)_{k=1}^n \\ &\|\tilde{g}_1\|\beta, \Sigma_{S_0,R}{}^{(1-\alpha,\overline{\Sigma_{S_0,R}}\cap\overline{\Sigma_{\Gamma_w,R}})} \le C(\beta,\Lambda)\|g_1\|_{\beta,\Lambda}^{(1-\alpha,\partial\Lambda)} \\ &\|\tilde{g}_2\|_{\beta,\Sigma_{\Gamma_w,R}}^{(1-\alpha,\overline{\Sigma_{S_0,R}}\cap\overline{\Sigma_{\Gamma_w,R}})} \le C(\beta,\Lambda)\|g_2\|_{\beta,\Gamma_{w,r_s}}^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}. \end{aligned} \tag{B.13}$$

Let us set $d(y) := dist(y, \overline{\Sigma_{S_0,R}} \cap \overline{\Sigma_{\Gamma_w,R}})$ and

$$\begin{aligned} &M_1(R) := R\|\psi\|_{1,0,\mathcal{N}_{r_s}^+}^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})} + \|F\|_{0,\mathcal{N}_{r_s}^+}^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}, \\ &M_2(R) := M_1(R) + \|g_1\|_{0,\Lambda}^{(1-\alpha,\partial\Lambda)} + \|g_2\|_{0,\Sigma_{\Gamma_w,R}}^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}. \end{aligned} \tag{B.14}$$

Using (B.12), (B.13) and the continuity of $\tilde{a}_{kl}(y,0)$ in $\delta^*(y_0,y)$, we get, for any $y\in\mathcal{D}_{R_1}$,

$$\begin{aligned}&|h_1(y)|\le M_1(R)[d(y)]^{-1+\alpha},\\&||h_2(y)|-|w(y)||\le M_2(R)[d(y)]^{-1+\alpha},\quad |h_3(y)|\le M_2(R)[d(y)]^{-1+\alpha}.\end{aligned}\tag{B.15}$$

By the Poincaré inequality with scaling, and (B.15), for any $\varepsilon>0$, we have

$$\begin{aligned}&\int_{\Sigma_{S_0,R}}|h_2||w|dA_y\\&\le C(\varepsilon,n,\alpha)R[M_2(R)]^2\int_{\Sigma_{S_0,R}}[d(y)]^{2(-1+\alpha)}dA_y+(\varepsilon+C(n)R)\int_{\mathcal{D}_R}|Dw|^2dy.\end{aligned}\tag{B.16}$$

By (B.6), for any $R\le R_1$, $\Sigma_{S_0,R}$ satisfies

$$\Sigma_{S_0,R}\subset\{(r,y_1',\cdots,y_{n-1}'):r=r_s,y_{n-1}'\in(0,R),(y_1',\cdots,y_{n-2}')\in B_R^{(n-2)}(0)\}\tag{B.17}$$

where $B_R^k(0)$ is a ball in $\mathbb{R}^k$ of radius R centered at $0\in\mathbb{R}^k$. This provides

$$\int_{\Sigma_{S_0,R}}[d(y)]^{2(-1+\alpha)}dA_y\le C(n)vol(B_R^{(n-2)}(0))\int_0^R t^{2(-1+\alpha)}dt\le C(n,\alpha)R^{n-3+2\alpha}.\tag{B.18}$$

Note that $\alpha>\frac{1}{2}$ is an important condition in (B.18). By (B.16) and (B.18), we obtain

$$\int_{\Sigma_{S_0,R}}|h_2||w|dA_y\le C(\varepsilon,n,\alpha)R^{n-2+2\alpha}[M_2(R)]^2+(\varepsilon+C(n)R)\int_{\mathcal{D}_R}|Dw|^2dy.$$

One can treat $\int_{\mathcal{D}_R}|h_1||Dw|dy,\int_{\Sigma_{\Gamma_w,R}}|h_3||w|dA_y$ similarly, and combine all the results together so that (B.11) implies

$$\lambda\int_{\mathcal{D}_R}|Dw|^2dy\le C(\varepsilon,n,\alpha)R^{n-2+2\alpha}[M_2(R)]^2+(\varepsilon+C(n)R)\int_{\mathcal{D}_R}|Dw|^2dy.\tag{B.19}$$

Choose $\varepsilon=\frac{\lambda}{10}$, and reduce R_1 if necessary so that (B.19) implies

$$\int_{\mathcal{D}_R}|Dw|^2dy\le C(n,\alpha,\lambda)R^{n-2+2\alpha}[M_2(R)]^2\ \text{ for all }\ R\in(0,R_1].\tag{B.20}$$

Step 6. By (B.9) and (B.20), whenever $0<\varrho\le R\le R_1$, ϕ satisfies

$$\int_{\mathcal{D}_\varrho}|D\phi|^2dy\le C\big((\frac{\varrho}{R})^n\int_{\mathcal{D}_R}|D\phi|^2dy+R^{n-2+2\alpha}[M_2(R)]^2\big)$$

So, by [HL, Lemma 3.4] and (B.13), we obtain

$$\begin{aligned}\int_{\mathcal{D}_R}|D\phi|^2dy&\le CR^{n-2+2\alpha}(\frac{R^{2-2\alpha}}{R_1^n}\|D\phi\|^2_{L^2(\mathcal{D}_R)}+[M_2(R)]^2)\\&\le CR^{n-2+2\alpha}\big((\|\psi\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,0,\mathcal{N}^+_{r_s}})^2+[M_2(R_1)]^2\big).\end{aligned}\tag{B.21}$$

We note that the estimate (B.21) holds true for any choice of $x_0=h^{-1}(y_0)\in\mathcal{N}^+_{r_s}$ and any $R\in(0,R_1]$, moreover R_1 is uniform for all $x_0\in\mathcal{N}^+_{r_s}$. Since Λ is compact, and $\partial\Lambda$ is smooth, we can take a constant $R^*>0$ depending on Λ so that, by (B.21), the solution ψ to (6.7)-(6.10) satisfies

$$\int_{B_R(x_0)\cap\mathcal{N}^+_{r_s}}|D\psi|^2\le CR^{n-2+2\alpha}(\|\psi\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,0,\mathcal{N}^+_{r_s}}+K_0)^2\tag{B.22}$$

with $K_0 = \|F\|^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{0,\mathcal{N}^+_{r_s}} + \sum_{l=1,3}\|g_l\|^{(1-\alpha,\partial\Lambda)}_{0,\Lambda} + \|g_2\|^{(1-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{0,\Gamma_{w,r_s}}$.

By the interpolation inequality in [GT, Lemma 6.34], for any $\varepsilon > 0$, we have

$$\|\psi\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,0,\mathcal{N}^+_{r_s}} \leq C(\varepsilon,\alpha)|\psi|_{0,\mathcal{N}^+_{r_s}} + \varepsilon\|\psi\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,\alpha,\mathcal{N}^+_{r_s}}.$$

So, assuming $R^* < 1$ without loss of generality, by (B.3), (B.5), we obtain

$$\|\psi\|^{(-\alpha,\partial S_0\cup\partial\Gamma_{ex})}_{1,\alpha,\mathcal{N}^+_{r_s}} \leq C(|\psi|_{0,\mathcal{N}^+_{r_s}} + K(F,g_1,g_2,g_3))$$

where $K(F,g_1,g_2,g_3)$ is defined after (B.3).

Since $\mu_0 > 0$ in (6.8) by Lemma 2.5, by the uniqueness of solution([LI1, Corollary 2.5]) for (6.7)-(6.10), we have $|\psi|_{0,\mathcal{N}^+_{r_s}} \leq CK(F,g_1,g_2,g_3)$. The proof is complete. □

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